STEP-STRESS ACCELERATED LIFE TESTS - A PROPORTIONAL HAZARDS BASED NONPARAMETRIC MODEL

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ABSTRACT

Using data from a simple Step-Stress Accelerated Life Testing (SSALT), a nonparametric Proportional Hazards (PH) model is proposed for obtaining Upper Confidence Bounds (UCBs) for the cumulative failure probability of a product under normal use conditions. The approach is nonparametric in the sense that most of the functions involved in the model do not assume any specific forms, except for certain verifiable conditions. Test statistics are introduced to verify assumptions about the model and to test the Goodness of Fit (GOF) of the proposed model to the data. A numerical example, using data simulated from the lifetime distribution of an existing parametric study on Metal-Oxide-Semiconductor capacitors, is used to illustrate the proposed methods. Discussions on how to determining the optimal stress levels and sample size are also given.

Keywords: Lifetime distribution; Accelerated life test; Step-stress tests; Nonparametric approach; Proportional hazards model; Cumulative exposure; Goodness-of-fit test; Optimal design.

1. INTRODUCTION

The lifetime distribution of a product is often used to assess the cumulative likelihood of product failure up to a certain time. Such information is important to managers when, for example, establishing a warranty period or when pricing extended warranties. To obtain an estimate of the lifetime distribution of a product in a timely manner, Accelerated Life Testing (ALT) is widely used in industry. In an ALT, test units are exposed to relatively greater stress with the objective of increasing the likelihood of observing failures. Inferences about the true lifetime distribution under normal use conditions can then be made based on these observed accelerated failure times.

Different stress strategies are used when conducting an ALT. For example, in a Parallel Constant-Stress ALT (PCSALT), test units are divided into groups and each group is tested under a different stress level throughout the test. Another strategy is the Step-Stress ALT (SSALT). In a simple SSALT, the test begins by exposing all test units to a pre-specified stress level. For those test units that survive up to a pre-specified time, the stress level is changed and held constant until all units fail (uncensored test) or until a pre-specified test termination time is attained (censored test).

SSALTs are commonly used; for example, in testing diodes, electrical cable insulation, insulating fluid (Bai and Chun (1991)), rear suspension aft lateral link (Lu, Leiman, Rudy, and Lee (2006)), etc. Possible advantages of using SSALTs include:

(i)*Time saving*: A SSALT can substantially shorten the duration of the test without affecting the accuracy of lifetime distribution estimates (Zhao and Elsayed (2005)).

(ii) *Practical*: Relative to a PCSALT, a SSALT is more practical in the sense that fewer test units are required (Tseng and Wen (2000)).

(iii) *Adaptable*: A SSALT is a flexible test strategy, especially for new products when one presumably has little information regarding appropriate test stresses. In a SSALT, the later stress levels, as well as the transition times, could be dynamically adjusted as failure information is being gathered over time.

(iv) *Economical*: Testing units at higher stress levels often incurs greater costs. Therefore, testing all units at a lower stress level at the beginning of a SSALT may lower the expected cost. For example, Tang, Yang, and Xie (2004) proposed an optimal step-stress accelerated degradation plan that requires fewer samples and less test time, and hence reduces the total test cost, than an existing degradation test.

Khamis (1997) studied the SSALT and PCSALT, assuming a Weibull lifetime distribution, and concluded that SSALT should be considered as an alternative to the PCSALT when there is heavy censoring for the PCSALT at lower stress levels. Ma and Meeker (2008) concluded that a SSALT can have a smaller variance for the Maximum Likelihood Estimates (MLEs) of the quantiles of a lifetime distribution than the PCSALT when the scale parameter of the assumed lifetime distribution (Weibull or lognormal) is large or when the failure probability at the censored time under high stress levels is relatively small.

1.1. Parametric and Nonparametric Models

In a parametric analysis for a SSALT, one assumes a true lifetime distribution (e.g., exponential, Weibull, or lognormal) with certain unknown parameters and an acceleration function to relate these parameters to stress levels (Zhao and Elsayed (2005)). Other assumptions such as Nelson's (1980) Cumulative Exposure (CE) assumption (being described later) are used for deriving the accelerated lifetime distribution for the data obtained from a SSALT, to obtain sample estimates of the true lifetime parameters. Tang, Sun, Goh, and Ong (1996) proposed a linear CE model and obtained the MLEs of the lifetime parameters in a multi-censored SSALT. Balakrishnan and Han (2008) and Han and Balakrishnan (2010) considered an exponential SSALT with competing risks. For more recent studies regarding parametric inferential methods of SSALT, see, for example, Balakrishnan, Zhang, and Xie (2009), Balakrishnan, Beutner, and Kateri (2009), Wang (2010), and Lee and Pan (2010).

As to optimally designing a SSALT, Ismail and Aly (2009) and Srivastava and Mittal (2010) considered the optimum plan for a *partial* SSALT. Most of the literature assumed Nelson's CE assumption with different lifetime distributions and censoring schemes. For example, Miller and Nelson (1983), Bai and Chun (1991), Bai and Kim (1993), Alhadeed and Yang (2005), Elsayed and Zhang (2005), and Gouno (2007). Since the parametric approach is not the main focus of the paper, the reader is referred to Nelson (1990, 2005a, 2005b) for comprehensive overviews on theory of parametric PCSALT and SSALT.

Although specific parametric lifetime distributions are used for certain types of products, Nelson (1990) pointed out that the assumed form of the lifetime distribution is questionable for many products (also see Pascula and Montepiedra (2003)). Moreover, when assessing the ability of several potential lifetime models, it can be difficult to assess which one provides a better fit to the data. Significant errors in the extrapolation of ALT results could occur if the assumed model does not provide a good approximation of the actual failure mechanism. Thus, a nonparametric model absent specific assumptions regarding lifetime distributions should be considered.

A few research efforts have considered the nonparametric approach for analyzing failure time data under a PCSALT (Schmoyer (1986)). In particular, for a PCSALT with two parallel constant-stress levels, Schmoyer (1988, 1991) suggested a nonparametric estimation procedure for obtaining an Upper Confidence Bound (UCB) for the cumulative failure probability at any given point under normal use conditions. Some nonparametric approaches for SSALT have been proposed, including Shaked and Singpurwalla (1983) and Tyoskin and Krivolapov (1996).

1.2. Objective and Organization of the Paper

Given the advantages of SSALT and the lack of a nonparametric model for data analysis, we propose a nonparametric approach for estimating the true lifetime Cumulative Distribution Function (CDF) based on data obtained from a simple *step-up* SSALT, where the stress level is switched to a high level at certain stress change time.

The paper is organized as follows. In Section 2, a Proportional Hazards (PH) based nonparametric model is introduced for the underlying lifetime distribution. The CE assumption of Nelson (1980) is also used, in lieu of acceleration functions common to the parametric approach, to derive the lifetime CDF under a simple SSALT. Using data from this CDF, we obtain in Section 3 a nonparametric UCB for the lifetime CDF under normal use conditions. Section 4 provides methods for verifying the sufficient conditions in the development of the UCB in Section 3. Section 5 develops a procedure for assessing the Goodness of Fit (GOF) of the underlying PH model to the data. A simulated numerical study utilizing the work of Elsayed and Zhang (2005) regarding Metal-Oxide-Semiconductor capacitors is provided in Section 6 to both illustrate the implementation and to empirically assess the sensitivity of the proposed method to various stress levels and sample size. Discussions on how to determine the optimal stress levels and sample size are also given. Finally, Section 7 summarizes the results and suggests directions for future research.

2. A PH-BASED NONPARAMETRIC MODEL

In this section, a nonparametric PH model for the lifetime CDF is introduced. Then, with the CE assumption, we derive the lifetime CDF under a simple SSALT.

2.1. The PH Model

Consistent with the work of Schmoyer (1991), the lifetime CDF for a given constant-stress is assumed to follow a general PH model. A PH model assumes that the effect of stress is multiplicative in the hazard rate and that the ratio of hazard rates under two different stress levels is constant over time. That is, if a unit is tested under stress level x throughout the test, the hazard rate function at any given time t is

$$\lambda(t;x) = g(x)h'(t)\,, \quad t \geq 0 \qquad (1)$$

where

$g(x) \equiv$ the damage rate function which is a non-negative and non-decreasing sigmoid function of stress defined on $[0, \infty)$;

$h(t) \equiv$ a non-negative, non-decreasing, and differentiable function of time.

A non-negative and non-decreasing function is sigmoid if there is a point $M \in [0, \infty)$ to the left of which the function is convex and to the right concave. (The cases $M = 0$ or $M = \infty$, i.e., concave or convex, are admitted). Several commonly used damage rate functions in parametric models conform to such assumptions (see examples in Schmoyer (1988, 1991)).

Let x_0 be the normal use stress level and x_1 and x_2 ($x_0 \le x_1 < x_2$) be two test stress levels, which are assumed to be pre-determined and fixed. The hazard functions for units under these two different test stress levels are proportional since $\lambda(t; x_1)/\lambda(t; x_2) = g(x_1)/g(x_2)$ is constant for all $t \ge 0$. The model in (1) is nonparametric because we do not specify any parametric forms of the functions $g(x)$ and $h(t)$, except certain verifiable conditions to be described later. It follows from (1) that the lifetime CDF under a constant-stress x_i ($i = 1, 2$) is

$$F_i(t) = 1 - \exp(-g(x_i)h(t)), \quad t \ge 0. \tag{2}$$

2.2. Lifetime CDF Under a Simple SSALT

Let τ be the stress change time (from x_1 to x_2) in a simple SSALT. To derive the lifetime CDF under the simple SSALT, a model is required to relate this CDF to the two CDFs in (2) under constant x_1 and x_2, respectively. For this purpose, Nelson's (1980) CE assumption is used. This assumption assumes that the remaining life of a test unit at any given time depends only on the cumulative exposure it has received up to that time and the current stress level, regardless of how the exposure was accumulated.

Under the CE assumption, there exists a time, denoted by $s\ (<\tau)$, such that the cumulative amount of exposure at τ under stress level x_1 is equivalent to the CE amount at s as if the ALT had been conducted with stress level x_2 (see Figure 1). Since the CEs are equivalent, we have $F_2(s) = F_1(\tau)$, and hence $g(x_2)h(s) = g(x_1)h(\tau)$. Therefore, the lifetime CDF of a test unit under a simple SSALT is

$$F_{SS}(t) = \begin{cases} F_1(t) = 1 - \exp(-g(x_1)h(t)), & \text{if } t \le \tau \\ F_2(t-\tau+s) = 1 - \exp(-g(x_2)h(t-\tau+s)), & \text{if } t > \tau \end{cases} \tag{3}$$

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Figure 1

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3. UCBs OF LIFETIME CDF UNDER NORMAL CONDITIONS

A nonparametric UCB, based on data obtained from a simple SSALT, is developed for the cumulative failure probability under normal use conditions, x_0. The UCB is derived with the following additional assumption on $g(\cdot)$ (a method for verifying this assumption will be proposed in Section 4):

$$\frac{g(x_0)}{x_0} \le \frac{g(x_1)}{x_1}, \quad \text{or} \quad \frac{g(x_0)}{g(x_1)} \le \frac{x_0}{x_1}. \tag{4}$$

Denote $\Pr(t \mid x_i) \equiv F_i(t)$ and $\Pr(t \mid SS) \equiv F_{SS}(t)$. Then, using (4) and the fact that $\Pr(t \mid SS)$ $= \Pr(t \mid x_1)$, for $t \le \tau$, we can prove

Proposition 1. Under model (3) and assumption (4), the following inequality holds:

$$\Pr(t \mid x_0) \le 1 - \left[1 - \Pr(t \mid SS)\right]^{x_0/x_1}, \quad t \ge 0. \tag{5}$$

Let $\hat{\Pr}(t \mid SS)$ be the empirical CDF for $F_{SS}(t)$, a conservative $100(1-\alpha)\%$ UCB of $\Pr(t \mid SS)$ in (5) is (Meeker and Escobar (1998))

$$\tilde{\Pr}_{\alpha}(t \mid SS) = \left\{ 1 + \frac{n - n\hat{\Pr}(t \mid SS)}{(n\hat{\Pr}(t \mid SS)+1) F_{(1-\alpha; 2n\hat{\Pr}(t \mid SS)+2; 2n-2n\hat{\Pr}(t \mid SS))}} \right\}^{-1}, \tag{6}$$

where n is the total number of test units, and $F_{(p;\, v_1;\, v_2)}$ is the p-th percentile of the F distribution with (v_1, v_2) degrees of freedom. Then by (5), $1 - \left[1 - \tilde{\Pr}_{0.05}(t \mid SS)\right]^{x_0/x_1}$ provides a 95% UCB for $\Pr(t \mid x_0)$.

4. VERIFICATION OF ASSUMPTION (4)

A method for verifying assumption (4) is provided in Section 4.1. To simplify the notation, denote $c \equiv g(x_1)/g(x_2)$, which is a constant since x_1 and x_2 are fixed. Then $c \in (0,1)$. Estimation procedures for c are given in Sections 4.2-4.4.

4.1. Verification Procedure and Unimodality

Since the damage rate function $g(x)$ is assumed to be sigmoid, the function $g(x)/x$ is unimodal (Schmoyer (1988)). Non-increasing and non-decreasing functions are considered unimodal functions (with infinite mode). Assume that

$$\frac{g(x_1)}{x_1} \le \frac{g(x_2)}{x_2}, \quad \text{or equivalently} \quad c \le \frac{x_1}{x_2}, \tag{7}$$

then $g(x)/x$ cannot be non-increasing over $(0, \infty)$. First, if $g(x)/x$ is non-decreasing, then assumption (4) is true for $0 < x_0 \le x_1$. On the other hand, if $g(x)/x$ has a finite mode, then x_1 has to be to the left of the mode because of (7). This implies that $g(x)/x$ is non-decreasing in the interval $[0, x_1]$, and hence assumption (4) is true for $0 < x_0 \le x_1$. Consequently, if we can verify (7), then (4) is true.

According to (3) and the definition of s, we have

$$\frac{-ln(1-\Pr(t \mid SS))}{-ln(1-\Pr(t-s+\tau \mid SS))} = c \text{, for all } t \in [s, \tau]. \tag{8}$$

An UCB of the left hand side of (8) can be obtained as the ratio of a $100(1-\alpha)^{1/2}\%$ UCB of $\Pr(t \mid SS)$ to a $100(1-\alpha)^{1/2}\%$ lower confidence bound of $\Pr(t-s+\tau \mid SS)$. If this UCB for the left hand side of (8) is smaller than x_1 / x_2, we conclude that (7) is valid up to a certain level of confidence. Schmoyer (1991) used a similar procedure for a PCSALT. However, unlike Schmoyer (1991), the range of useful *t* for a simple SSALT is limited to the interval $[s, \tau]$, where s as well as c are unknown parameters to be estimated.

4.2. Modified Likelihood Function of Parameter *c*

We propose a modified likelihood function for obtaining an estimate of *c*. This likelihood function will also be used later for developing a GOF test of the PH model to the data.

Suppose that the parameter *s* is known or well-estimated. Let $t_{11}, t_{12}, \ldots, t_{1,n_1}$ be the observed failure times (under the stress level x_1) within the interval $[s, \tau]$ and $t_{21}, t_{22}, \ldots, t_{2,n_2}$ be the observed failure times (under the stress level x_2) after time τ. Let $t'_{2j} \equiv s + t_{2j} - \tau$, $j = 1, \ldots, n_2$, be the shifted failure times. Under model (3) and Figure 1, $\{t_{11}, t_{12}, \ldots, t_{1,n_1}\}$ and $\{t'_{21}, t'_{22}, \ldots, t'_{2,n_2}\}$ are considered as two samples from the two CDFs under x_1 and x_2, respectively, over the same interval $[s, \tau]$. Note that the PH model in (3) also holds for any monotone transformation of the failure times in the sense that the distributional form in (3) stays the same. Hence, for the estimation (and hypothesis testing) problem about the model unknown parameters to remain invariant under these time transformations, the estimators (and test statistic) will depend only on the ranking information of $t_{11}, t_{12}, \dots, t_{1,n_1}$ and $t'_{21}, t'_{22}, \dots, t'_{2,n_2}$ (see Kalbfleisch and Prentice (1980, Chapter 4) and Schmoyer (1991)). The joint ranking information is characterized by $(a_0,\ a_1,\ \dots,\ a_{n_1})$, where a_i is the number of t'_{2j} 's such that $t_{1i} \le t'_{2j} < t_{1,i+1}$ with $t_{10} \equiv s$ and $t_{1,n_1+1} \equiv \infty$, for $i = 0,\ 1,\ \dots,\ n_1$.

Let $R_{it} \equiv \exp(-g(x_i)h(t))$, which is the survival probability of a test item at time t as if the item had been exposed to the constant stress level x_i throughout the test. The conditional likelihood function of c is (derivation in Appendix)

$$L(c \mid a_0, a_1, \dots, a_{n_1}, s, n_1, n_2) = \frac{n_1!\,n_2!}{a_{n_1}!}(R_{2s}c)^{n_1}\prod_{i=1}^{n_1}\prod_{k=0}^{a_{i-1}}\left(\frac{1}{\sum_{j=i}^{n_1} a_j + (n_1 - i + 1)c + k}\right), \qquad (9)$$

where R_{2s} is unknown. However, according to the CE assumption, $R_{2s} = R_{1\tau}$. Thus, the empirical survival probability, $\hat{R}_{1\tau} = 1 - \hat{\Pr}(\tau \mid SS)$ is used as an estimate of R_{2s} in (9) to obtain an approximate conditional likelihood function of c.

4.3. Search Procedure for Parameters *s* and *c*

It is apparent from equations (8) and (9) that estimation of s depends on c and vice versa. In this section, a search procedure is suggested for the simultaneous estimation of these two parameters. The proposed joint estimation procedure is iterative and based on the following considerations:

(a) For a given *c*, an estimate of *s* ($\in [0, \tau]$) is that satisfying (8);

(b) If s is given (estimated), the MLE of c can be obtained via (9).

The proposed search procedure is given as follows:

Step 1. Assume we have an initial estimate of c, say c_1 (a method for obtaining a good c_1 is provided later in this section). We re-parameterize s such that $s = \alpha\tau$, where $\alpha \in (0, 1)$. Let t' be the last failure time such that $\hat{\Pr}(t \mid SS) = 1$, for all $t \geq t'$. We seek an estimate of α (and hence of s) such that the following average squared error is minimized:

$$SV(\alpha, c_1) = \begin{cases} \dfrac{1}{t'-\tau} \displaystyle\int_{\alpha\tau}^{\alpha\tau+(t'-\tau)} (c(t,\alpha)-c_1)^2\,dt, & \text{if } t'-\tau \leq \tau-\alpha\tau \\ \dfrac{1}{\tau-\alpha\tau} \displaystyle\int_{\alpha\tau}^{\tau} (c(t,\alpha)-c_1)^2\,dt, & \text{if } t'-\tau > \tau-\alpha\tau \end{cases} \tag{10}$$

where (cf. (8))

$$c(t,\alpha) \equiv \frac{-ln(1-\hat{\Pr}(t \mid SS))}{-ln(1-\hat{\Pr}(t-\alpha\tau+\tau \mid SS))}, \quad t \in [\alpha\tau, \min(\alpha\tau + t' - \tau, \tau)].$$

The rationale for (10) is related to consideration (a) above. The minimizing α value is denoted by α_1 and the corresponding s_1 by $s_1 = \alpha_1\tau$.

Step 2. Using $s_1 = \alpha_1\tau$ as an estimate of s, we find c to maximize the likelihood function (9). The resulting estimate of c is denoted by c_2.

Step 3. Replace c_1 in Step 1 by c_2 and repeat these steps until the estimated values of (c, s) converge.

Initial Estimate of c

We propose the following procedure for obtaining a good initial estimate of c, c_1.

(i) The procedure begins by obtaining a reasonable estimate of s within the interval $(0, \tau)$. Recall from (8) that $c(t,\alpha)$, as a function of *t* and *α*, should be constant, for all $t \in (s,\tau)$. In other words, the most reasonable estimated value of $s\ (=\alpha\tau)$ would be the point which generates the most stable quantities of $c(t,\alpha)$ for all $t \in (\alpha\tau,\tau)$. Let $\overline{c}(\alpha)$ be the average of $c(t,\alpha)$ over $t \in (\alpha\tau,\tau)$. The following modified criterion of (10) provides a useful means for obtaining a reasonably good initial value of s or α:

$$SV\left(\alpha,\overline{c}(\alpha)\right)=\begin{cases}\dfrac{1}{t'-\tau}\displaystyle\int_{\alpha\tau}^{\alpha\tau+(t'-\tau)}\left(c(t,\alpha)-\overline{c}(\alpha)\right)^2 dt, & \text{if } t'-\tau \le \tau-\alpha\tau\\ \dfrac{1}{\tau-\alpha\tau}\displaystyle\int_{\alpha\tau}^{\tau}\left(c(t,\alpha)-\overline{c}(\alpha)\right)^2 dt, & \text{if } t'-\tau > \tau-\alpha\tau\end{cases} \tag{11}$$

(ii) Substitute the estimated $s \equiv \alpha\tau$ into (9) to obtain c_1 for Step 1 of the search process.

Note that taking logarithmetic transformations in (10) and (11) provides the following alternative criteria, which may be computationally more convenient:

$$SV'\left(\alpha,ln(c_1)\right)=\begin{cases}\dfrac{1}{t'-\tau}\displaystyle\int_{\alpha\tau}^{\alpha\tau+(t'-\tau)}\left(ln(c(t,\alpha))-ln(c_1)\right)^2 dt, & \text{if } t'-\tau \le \tau-\alpha\tau\\ \dfrac{1}{\tau-\alpha\tau}\displaystyle\int_{\alpha\tau}^{\tau}\left(ln(c(t,\alpha))-ln(c_1)\right)^2 dt, & \text{if } t'-\tau > \tau-\alpha\tau\end{cases} \tag{12}$$

and

$$SV'\left(\alpha,ln(\overline{c}(\alpha))\right)=\begin{cases}\dfrac{1}{t'-\tau}\displaystyle\int_{\alpha\tau}^{\alpha\tau+(t'-\tau)}\left(ln(c(t,\alpha))-ln(\overline{c}(\alpha))\right)^2 dt, & \text{if } t'-\tau \le \tau-\alpha\tau\\ \dfrac{1}{\tau-\alpha\tau}\displaystyle\int_{\alpha\tau}^{\tau}\left(ln(c(t,\alpha))-ln(\overline{c}(\alpha))\right)^2 dt, & \text{if } t'-\tau > \tau-\alpha\tau\end{cases} \tag{13}$$

4.4. Modified Likelihood Function of *c* for Type-I Censored SSALT

To obtain data in a timely manner, censoring is common in life testing. Consider a simple SSALT which is terminated at time η with some surviving items. Our search procedure for c

developed for the uncensored case in Section 4.3 can be directly applied to the censored SSALT; but the conditional likelihood function for c requires some modifications, as follows.

For a given s, two scenarios may occur following a transformation of the censored time from η to $\eta' \equiv s + (\eta - \tau)$.

<u>Scenario I</u>: $\eta' \geq \tau$ (see Figure 2)

As shown in the figure, after transforming the failure times under x_2, one could still count the number of transformed failure times within each interval $(t_{1i}, t_{1,i+1})$, $i = 0,1,\ldots,n_1$, to observe the joint ranks, $(a_0, a_1, \ldots, a_{n_1})$. Hence, the conditional likelihood function of c in (9) for the uncensored SSALT can be used directly for this scenario.

<u>Scenario II</u>: $\eta' < \tau$ (see Figure 3)

First, let t_{1m} be the last observed failure time under x_1 before η'. Under this scenario, one would not be able to obtain the exact number of transformed failure times falling within each interval $(t_{1i}, t_{1,i+1})$, for $i \geq m$. Thus, define a_m as the number of surviving items plus the number of transformed failure times under stress x_2 that exceed t_{1m}. Instead of $(a_0, a_1, \ldots, a_{n_1})$, one would only observe $(a_0, a_1, \ldots, a_m)$ and the conditional likelihood function of c based on $(a_0, a_1, \ldots, a_m)$ is

$$L(c \mid a_0, a_1, \ldots, a_m, m, n_2, s) = \frac{m! n_2!}{a_m!} (R_{2s} c)^m \prod_{i=1}^{m} \prod_{k=0}^{a_{i-1}} \left(\frac{1}{\sum_{j=i}^{m} a_j + (m-i+1)c + k} \right). \tag{14}$$

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Figures 2 and 3

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To estimate parameters c and s under type-I censoring, we replace t' with the censored time η in

Step 1 in Section 4.3 and use the likelihood function (14) instead of (9) in Step 2 if Scenario II occurs.

5. GOF Test of PH Model

In this section we first derive a GOF Likelihood Ratio Test (LRT) for the assumed PH model when the test is uncensored. We then extend the proposed methodology to accommodate Type-I censoring schemes in Section 5.2.

5.1. GOF for Uncensored SSALT

Under the PH assumptions, if we have an uncensored SSALT, then the joint ranks $(a_0, a_1, \ldots, a_{n_1})$ are observed and the constrained likelihood function based on the joint ranks $(a_0, a_1, \ldots, a_{n_1})$ is provided in (9). Without the PH assumptions, the unconstrained conditional likelihood of $(a_0, a_1, \ldots a_{n_1})$ is (see Schmoyer (1991))

$$\binom{n_2}{a_0, a_1, \ldots, a_{n_1}} \prod_{i=0}^{n_1} \left(\frac{a_i}{n_2} \right)^{a_i} . \tag{15}$$

The LRT statistic for the GOF test of the PH model is then the ratio of (9) to (15).

The exact null distribution of the LRT statistic, when the joint ranks $(a_0, a_1, ..., a_{n_1})$ are considered random, is difficult to obtain analytically. However, via simulation of $(a_0, a_1, ..., a_{n_1})$, we can obtain an observed significance level (i.e. *p*-value) for the current observed value of the LRT statistic. Since $(a_0, a_1, ..., a_{n_1})$ is invariant under any monotone transformation of the failure times, the distribution of the LRT statistic is therefore independent of $h(\cdot)$ in the PH model and depends on $g(x_1)$ and $g(x_2)$ only through $c = g(x_1)/g(x_2)$. However, unlike Schmoyer's (1991) approach in which two constant stress levels are applied in parallel over time $t \in [0, \infty)$, the interval for comparison in a simple SSALT is limited to $t \in (s, \tau)$. Therefore, the Monte Carlo simulation approach of Schmoyer (1991) is revised for the simple SSALT, as follows.

(i) Generate n_1 pseudorandom failure times, $t_{11}, t_{12}, \ldots, t_{1,n_1}$, from the $\frac{1}{\hat{c}} Exp(1)$ distribution (where $\hat{c}$ is the final estimate of c obtained from Section 4) satisfying

$$\hat{\Pr}(s \mid SS)\text{-th percentile of } \frac{1}{\hat{c}} Exp(1) \le t_{1i} \le \hat{\Pr}(\tau \mid SS)\text{-th percentile of } \frac{1}{\hat{c}} Exp(1).$$

(ii) Generate n_2 pseudorandom failure times, $t'_{21}, t'_{22}, \ldots, t'_{2,n_2}$, from $Exp(1)$, conditioned on

$$\hat{\Pr}(\tau \mid SS)\text{-th percentile of } Exp(1) \le t'_{2j}.$$

(iii) Based on the simulated failure times from (i) and (ii), the values of $(a_0, a_1, \ldots, a_{n_1})$ and the likelihood ratio are obtained.

Repeat (i) to (iii) many times (1000 is sufficient). The significance level (p-value) for the LRT statistic from the data is then the percentile of the observed value of the likelihood ratio relative to the simulated likelihood ratios. Hence, the decision is to reject the null hypothesis, H_0: PH model is adequate, at $\alpha = 5\%$ significance level if

$$p\text{-value} \equiv \frac{\#\{\text{simulated ratio} \ge \text{the ratio from data}\}}{1000} \le 0.05.$$

5.2. GOF Test for Type-I Censored SSALT

For a Type-I censored SSALT, the previous GOF test procedures in Section 4.4 require some minor modifications but only for Scenario II, because under Scenario I we still observe the joint ranks $(a_0, a_1, \ldots, a_{n_1})$ here. The modifications are: Replace n_1 with m and τ with η' in (i), and replace $(a_0, a_1, \ldots, a_{n_1})$ with $(a_0, a_1, \ldots, a_m)$ in (iii).

6. NUMERICAL EXAMPLE

6.1. Underlying Model and Data Generation

Elsayed and Zhang (2005) introduced an example in which a simple SSALT plan was designed for Metal-Oxide-Semiconductor capacitors to estimate the reliability distribution at designed temperature, $x_0 = 50^{\mathrm{o}}\mathrm{C}$. In their parametric model, the lifetime CDF in (3) is

$$F_{SS}(t) = \begin{cases} 1-\exp\left\{-\left[e^{-\frac{\beta}{273.16+x_1}}(\gamma_0 t+\gamma_1 t^2/2)\right]\right\}, & \text{if } t \le \tau \\ 1-\exp\left\{-\left[e^{-\frac{\beta}{273.16+x_2}}(\gamma_0(t-\tau+s)+\gamma_1(t-\tau+s)^2/2)\right]\right\}, & \text{if } t > \tau \end{cases}$$

with $\beta = 3800$, $\gamma_0 = 0.0001$, and $\gamma_1 = 0.55$. Here $g(x) = e^{-\frac{\beta}{273.16+x}}$ and $h(t) = \gamma_0 t + (\gamma_1/2)t^2$ in our notation. The optimal simple SSALT plan given by Elsayed and Zhang (2005) is: $x_1 = 145^{\mathrm{o}}C$, $x_2 = 250^{\mathrm{o}}C$, and $\tau = 262.5$ hours. Since s satisfies $F_1(\tau) = F_2(s)$, we have $s = 105.46 = 0.4\,\tau$ and $c = e^{-\frac{3800}{273.16+145}} \Big/ e^{-\frac{3800}{273.16+250}} = 0.1614$. Using these models and parameters, a total of $n = 200$ failure times (assumed in Elsayed and Zhang (2005)) are generated from $F_{SS}(\cdot)$ to obtain the corresponding empirical CDF, $\hat{\Pr}(t \mid SS)$, needed in our analysis.

6.2. Estimations of Parameters *c* and *s*

To estimate s and c using the iterative search procedures developed in Section 4.3, we need initial estimates of these two parameters. All results are obtained by using Matlab.

Initial Parameter Estimates

To obtain c_1 for Step 1 of Section 4.3, criterion (13) was first used to obtain a reasonable estimate of α (and hence s). The plot of $SV'\left(\alpha, ln(\overline{c}(\alpha))\right)$ in (13) against α shows the minimum occurs at α = 0.439, and hence $s \equiv \alpha\tau$ =115.237. This value is then substituted into (9) to obtain the first conditional MLE of c, c_1 = 0.177. The estimates of (c, s) in the following 3 search iterations are: (0.177, 123.38), (0.168, 120.75), and (0.165, 120.75), respectively. Recall the actual values of s and c are 105.46 and 0.1614, respectively. Hence, the search procedure provided reasonable estimates of the parameters. The final values, (c, s) = (0.165, 120.75), will be used in the following numerical analysis.

Sensitivity Analysis of the Initial Value of c_1

Because of the random nature of data, we chose the following initial values of c_1 to assess the sensitivity of the search results to $c_1 : c_1 = 0.177 \pm 10\%(0.177)$, $c_1 = 0.177 \pm 20\%(0.177)$, $c_1 = 0.177$ $\pm\, 30\%(0.177)$, and $c_1 = 0.177 + 100\%(0.177)$. The resulting convergent estimates of (c, s) are shown in Table 1.

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Table 1
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From the results in Table 1, parameter estimates with a reasonably good initial estimate for c_1 tend to converge to the same final estimates. However, when the initial c_1 is substantially biased, the search procedure can produce biased results (the last two cases in Table 1). This suggests the importance of an appropriate initial estimate of c_1.

6.3. GOF Test of PH Model

The GOF test of the PH model for $F_i(t)$, $i = 1, 2,$ is conducted following the procedure proposed in Section 5, the parameter estimates used to implement the test are listed below:

$(c, s) = (0.165, 120.75)$, $n_1 = 109$, $n_2 = 23$, $\hat{\Pr}(s \mid SS) = 0.34$, and $\hat{\Pr}(\tau \mid SS) = 0.885$.

1000 samples are simulated and the corresponding likelihood ratios are calculated. The resulting *p*-value of the observed data is 0.477, and hence the PH model assumption is not rejected at $\alpha = 5\%$ significance level.

6.4. Data Analysis - Verification of (7)

Recall that Proposition 1 is used to construct UCBs for the lifetime CDF at different times, provided that (4) is true. Further, a sufficient condition for (4) is provided by (7). Two methods are introduced for verifying (7): one is based on Wilks' likelihood ratio statistic and the other is based on the upper bound approach developed in Section 4.

Likelihood Ratio Approach (Method 1)

To assess whether $c \le x_1/x_2$ in (7) is true, we first use Wilks' LRT statistic, defined by $R(c) = L(c)/L(\hat{c})$, where $L(c)$ is given in (9), to obtain a $100(1-\alpha)\%$ UCB of *c*. Since $-2\,ln(R(c))$ is approximately distributed as a Chi-square with one degree of freedom, an approximate $100(1-\alpha)\%$ likelihood based confidence interval for *c* is the set of all *c*'s satisfying

$$-2\,ln(R(c)) \le \chi_1^2(1-\alpha), \tag{16}$$

from which an UCB of *c* can be obtained. The plot of $-2\,ln(R(c))$ against *c* give an approximate 95% confidence interval (0.10, 0.28) with an UCB of c at 0.28, which is less than the ratio $x_1/x_2 = 0.58$. Condition (7) and hence (4) is verified with 95% confidence.

Upper Bound Method of Section 4.1 (Method 2)

The upper bound method of Section 4.1 uses equation (8). That is, if we can find an UCB of the left side of (8), using the empirical CDF, then we have an UCB for c. More specifically, using this method, any $t \in [s, \tau]$= [120.75, 262.5] could be selected and used to establish an UCB of c. We chose some values of *t* in this interval and all UCBs obtained were less than $x_1/x_2 = 0.58$ (figure is available from authors), suggesting that condition (7), and hence (4), is valid.

6.5. UCBs for CDF Under Normal Use Conditions

Since assumption (4) is valid, the nonparametric UCB of $P(t \mid x_0)$ can now be obtained from Proposition 1. Using (5) and (6) in Section 4, we obtain nonparametric UCBs of $\Pr(t \mid x_0)$ at $t = 100$, 150, 200, 250, 300, and 350 hours. For each *t*, the true probability value, the corresponding estimated nonparametric UCB, and the relative difference (bias) between the two are shown in Table 2.

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Table 2
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6.6. Experimental Analysis for Optimal Design of SSALT

We consider the design problem of SSALT under our current model setting. In traditional parametric ALT models, optimal stress levels are normally determined by minimizing asymptotic variances of unbiased estimates of (or functions of) parameters of a given lifetime distribution. However, this approach is not possible in a nonparametric setting because the lifetime distribution is not completely specified. Therefore, minimizing the Mean Squared Error (MSE) of estimated UCBs of lifetime CDF would provide a more appropriate criterion for the selection of the stress levels. Hence our approach uses a statistical experimental design with the following 4 factorial combinations of stress levels (x_1, x_2): $(55^{\circ}\mathrm{C}, 150^{\circ}\mathrm{C})$, $(55^{\circ}\mathrm{C}, 200^{\circ}\mathrm{C})$, $(100^{\circ}\mathrm{C}, 150^{\circ}\mathrm{C})$, and $(100^{\circ}\mathrm{C}, 200^{\circ}\mathrm{C})$. The low accelerated stress level is set at $x_1 = 55$°C, which is close to the normal level, $x_0 = 50$°C. We used the same stress change time as Elsayed and Zhang (2005): $\tau =$ 262.5 hours. For each stress combination, we simulated 100 simple SSALTs, each with a sample of 200 test units. The averages and standard errors of the 100 simulated estimated 95% UCBs (Equations (5) and (6)) at various t are given in Table 3. Using the MSE of our estimated UCB as the objective function, we conclude that x_1 should be kept as close to x_0 as possible before the stress change time τ. After τ, the second stress level, x_2, should also be kept as low as possible. This is true for any t. Further, as a part of the experimental analysis, we had graphed the main and interaction effects of the first and second stress level. Results suggest that the interaction effect between the two stress levels on the MSE is negligible.

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Table 3
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In addition, we have considered another design variable, namely, the sample size. A similar simulation study with several different sample sizes {50, 100, 200, 300, and 500} was conducted using $(x_1, x_2, \tau) = (145^{\circ}C, 250^{\circ}C, 262.5)$. The results are shown in Table 4. Both the mean and standard error of the estimated UCB at a given time appear to decrease as the sample size increases. That is, the estimated UCB becomes tighter and more precise as the sample size increases. In addition, before the stress change time, the standard error at a given time follows a typical inverse square-root type of relationship to the sample size (We also simulated a case with even lower stress levels, this relationship holds). The sample standard error becomes stable once the sample size reaches 200, which suggests a minimum sample size if one would like to obtain a more precise (smaller variance) estimated UCB of the lifetime CDF at use conditions.

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Table 4
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7. SUMMARY

In this paper we propose a nonparametric PH based model, combined with Nelson's (1980) CE assumption, for estimating the UCBs of the lifetime distribution under normal use conditions, via a simple SSALT. Procedures are also developed for testing the model assumptions. A parametric simple SSALT model was used to illustrate the proposed procedure. The results imply that the performance of the estimated UCBs was sensitive to the choices of the stress levels, x_1 and x_2. As expected, the closer these stress levels are to the normal stress level, the better the bounds performed. Such observations suggest that the design of a simple SSALT has critical effects on the inferences and should thereby be considered with care.

Notice that the proposed procedures are not applicable to another type of SSALT, the step-down SSALT, in which the stress level decreases over time. The step-down SSALT has important applications in practice because it can reveal certain "lurking degradation/failure modes" that are independent of the main observed failure mode at high stress or less accelerated by the stress variables being used (hence the CE assumption is not satisfied). Thus, the development of nonparametric analytic models for step-down SSALTs should be considered. Some other areas for future research include (1) extending this research from the PH model to the Accelerated Failure Time (AFT) model, and (2) the nonparametric analytical procedure for products that provides degradation data.

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Appendix. Derivation the Conditional Likelihood Function in (9)

We first consider the following case where, under both stress levels, data are left truncated at *s* but there is no right censoring time for both stress levels. Recall that $t_1, t_2, \ldots, t_{1,n_1}$ are the observed failure times (under the stress level x_1) within the interval $[s, \tau]$ and $R_{2s} = R_{1\tau} = e^{-g(x_1)h(\tau)}$ is the surviving probability under stress x_1 at time τ. Similar to Schmoyer (1991), we can derive the conditional likelihood function of parameter $c = g(x_1)/g(x_2)$ (conditional on *s*, n_1, and n_2), based on observations on $(a_0, a_1, \ldots, a_{n_1})$. Let $\Omega = \left\{ s \le t_{11} \le t_{12} \le \ldots \le t_{1,n_1} < \infty \right\}$. Then

$$\Pr(a_0, a_1, ..., a_{n_1} \mid s, n_1, n_2) = \int_\Omega \Pr(a_0, a_1, ..., a_{n_1} \mid s, n_1, n_2, t_{11}, t_{12}, ..., t_{1,n_1}) \prod_{i=1}^{n_1} dF_1(t_{1i})$$

$$= n_1! \int_\Omega \binom{n_2}{a_0, ..., a_{n_1}} \left(\frac{F_2(t_{11}) - F_2(s)}{R_{2s}} \right)^{a_0} \prod_{i=1}^{n_1 - 1} \left(\frac{F_2(t_{1,i+1}) - F_2(t_{1,i})}{R_{2s}} \right)^{a_i} \left(\frac{1 - F_2(t_{1,n_1})}{R_{2s}} \right)^{a_{n_1}}$$
$$\times \prod_{i=1}^{n_1} g(x_1) h'(t_{1i}) e^{-g(x_1)h(t_{1i})} dt_{1i}$$

Let $v_i = e^{-g(x_2)h(t_{1i})}$, the joint distribution above can be rewritten as

$$n_1! \binom{n_2}{a_0, ..., a_{n_1}} \left(\frac{1}{R_{2s}} \right)^{n_2} c^{n_1} \int_{\{0 \le v_{n_1} \le v_{n_1 - 1} \le \ldots \le v_1 \le R_{2s}\}} (R_{2s} - v_1)^{a_0} \prod_{i=1}^{n_1 - 1} (v_i - v_{i+1})^{a_i} (v_{n_1})^{a_{n_1}} \prod_{i=1}^{n_1} v_i^{c-1} dv_i \, .$$

Define w_j's by $v_i = R_{2s} \prod_{j=1}^{i} w_j$, $w_j \in [0,1]$ for all *j*, then the function above becomes

$$n_1! \binom{n_2}{a_0, ..., a_{n_1}} \left(\frac{1}{R_{2s}} \right)^{n_2} c^{n_1} \left(R_{2s} \right)^{n_1 + n_2} \prod_{i=1}^{n_1} \int_0^1 (1 - w_i)^{a_i - 1} (w_i)^{\sum_{j=i}^{n_1} a_j + c(n_1 - i + 1) - 1} dw_i \, ,$$

which can be simplified to (9). Note that, in a simple SSALT, the failure times under x_1 are right censored at τ. Now, when there is censoring under x_1 but not x_2, the joint ranks $(a_0, a_1, ..., a_{n_1})$ are still observed and hence, according to Schmoyer (1991), we could still use the conditional likelihood function of parameter *c* as above.

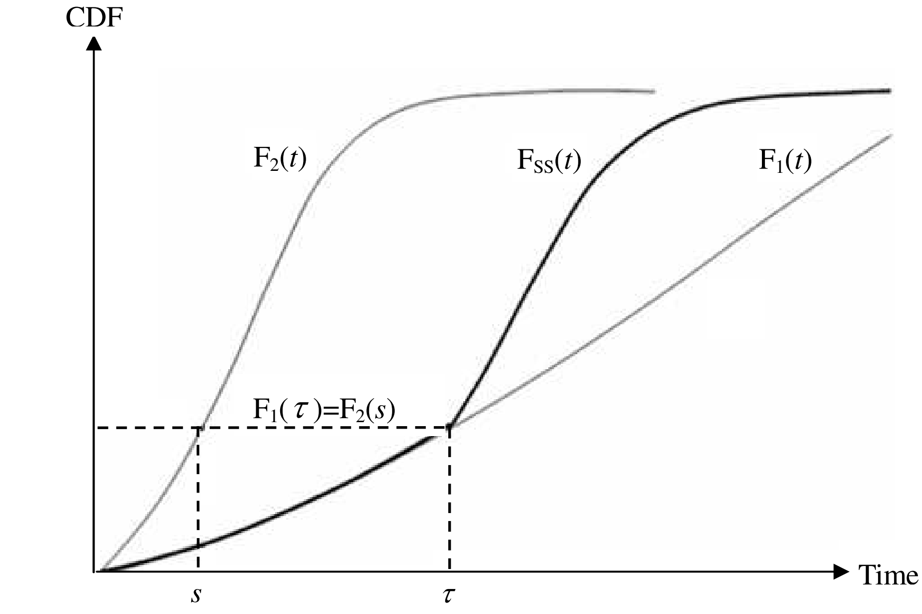

Figure 1. Lifetime CDF under Simple SSALT with Cumulative Exposure Assumption

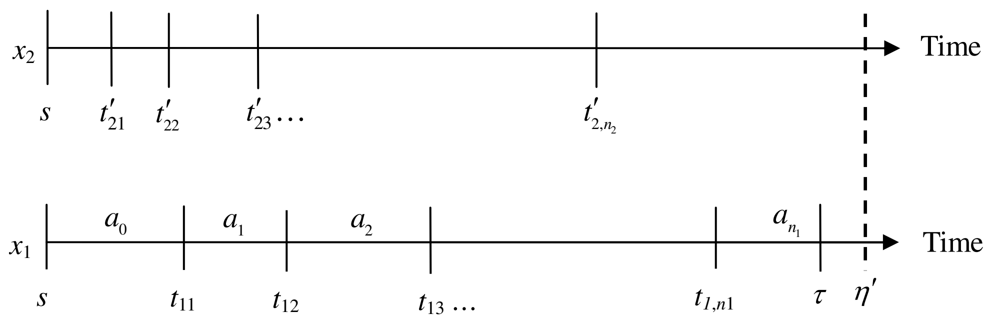

Figure 2. Joint Ranks $(a_1, a_2, \ldots a_{n_1})$ for Scenario I under Type-I Censoring

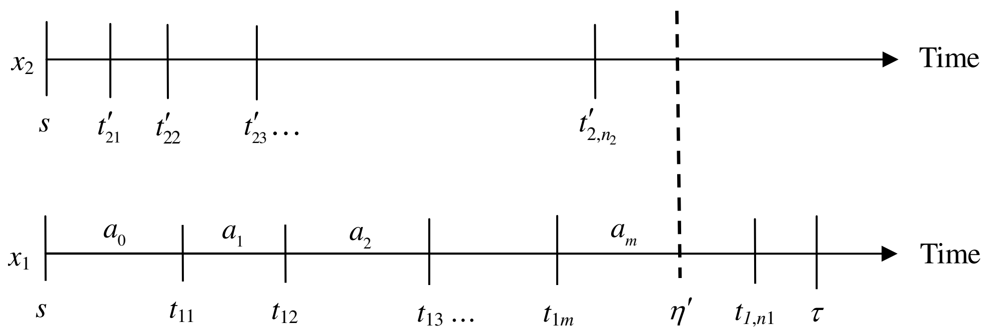

Figure 3. Joint Ranks $(a_1, a_2, \ldots a_m)$ for Scenario II under Type-I Censoring

Table 1. Sensitivity Analysis of (c, s) to the Initial Value c_1

c_1	*Final Estimates*
$0.177\times(1-10\%)$	$(c,s)=(0.165,120.75)$
$0.177\times(1+10\%)$	$(c,s)=(0.165,120.75)$
$0.177\times(1-20\%)$	$(c,s)=(0.165,120.75)$
$0.177\times(1+20\%)$	$(c,s)=(0.165,120.75)$
$0.177\times(1-30\%)$	$(c,s)=(0.165,120.75)$
$0.177\times(1+30\%)$	$(c,s)=(0.222,144.38)$
$0.177\times(1+100\%)$	$(c,s)=(0.300,168.00)$

Table 2. Estimated UCBs of CDFs at Different Times Under $(x_1,x_2)=(145^{o}C,250^{o}C)$.

Stress Levels (x_1,x_2)	*Time*	*True CDF*	*Estimated 95% Upper bound*	*Bias*
$(145^{o}C,250^{o}C)$	100	0.0213	0.1021	0.0808
	150	0.0472	0.2324	0.1852
	200	0.0824	0.3748	0.2924
	250	0.1258	0.5459	0.4201
	300	0.1760	0.8873	0.7113
	350	0.2316	1.0000	0.7684

Stress change time is τ = 262.5hrs and n = 200.

Table 3. The Averages and Variances of Estimated UCBs of CDFs at Different Times Under Different Stress Levels Combinations.

Stress Levels (x_1, x_2)	*Time*	*True CDF*	*Estimated 95% Upper bound Average (Standard error)*	*Bias*	*MSE*
$(55^oC, 150^oC)$	100	0.021274	0.044206 (0.013327)	0.022932	0.000703
	150	0.047232	0.079038 (0.015053)	0.031806	0.001238
	200	0.08242	0.123823 (0.014230)	0.041403	0.001917
	250	0.125759	0.176903 (0.007742)	0.051144	0.002676
	300	0.175959	0.366602 (0.024795)	0.190643	0.036960
	350	0.231582	0.607265 (0.028918)	0.375683	0.141974
$(55^oC, 200^oC)$	100	0.021274	0.047426 (0.012355)	0.026152	0.000837
	150	0.047232	0.083429 (0.014936)	0.036197	0.001533
	200	0.08242	0.126361 (0.013928)	0.043941	0.002125
	250	0.125759	0.177006 (0.007594)	0.051247	0.002684
	300	0.175959	0.480236 (0.028972)	0.304277	0.093424
	350	0.231582	0.810161 (0.025562)	0.578579	0.335407
$(100^oC, 150^oC)$	100	0.021274	0.073070 (0.011897)	0.051796	0.002824
	150	0.047232	0.139683 (0.014722)	0.092451	0.008764
	200	0.08242	0.225317 (0.018142)	0.142897	0.020749
	250	0.125759	0.321474 (0.008740)	0.195715	0.038381
	300	0.175959	0.488555 (0.020955)	0.312596	0.098155
	350	0.231582	0.654820 (0.028119)	0.423238	0.179921
$(100^oC, 250^oC)$	100	0.021274	0.073120 (0.013331)	0.051846	0.002866
	150	0.047232	0.138630 (0.015221)	0.091398	0.008585
	200	0.08242	0.222718 (0.014797)	0.140298	0.019902
	250	0.125759	0.319650 (0.009226)	0.193891	0.037679
	300	0.175959	0.567286 (0.026154)	0.391327	0.153821
	350	0.231582	0.813783 (0.033183)	0.582201	0.340060

Each of combination uses 100 replications each with sample size 200. The stress change time is $\tau = 262.5\text{hrs}$.

Table 4. Sampling Statistics of Estimated 95% UCBs of CDFs at Different Times and Different Sample Sizes.

Sample size	*Time*	*Average of estimated upper bounds*	*Standard error of estimated upper bounds*
50	100	0.156527	0.030644
	150	0.288459	0.040735
	200	0.447552	0.045764
	250	0.602557	0.029588
	300	0.914696	0.075995
	350	0.997205	0.015973
100	100	0.139803	0.020534
	150	0.267650	0.030159
	200	0.413278	0.030695
	250	0.558457	0.021348
	300	0.857066	0.064858
	350	0.995598	0.017511
200	100	0.126722	0.015132
	150	0.250260	0.019901
	200	0.394987	0.022479
	250	0.545360	0.015857
	300	0.839190	0.060806
	350	0.991334	0.020733
300	100	0.119654	0.012844
	150	0.241626	0.016922
	200	0.385172	0.017845
	250	0.533965	0.014134
	300	0.814647	0.039393
	350	0.989501	0.022712
500	100	0.117343	0.008949
	150	0.236356	0.012132
	200	0.378582	0.013519
	250	0.521545	0.00843
	300	0.794793	0.028789
	350	0.985406	0.025261

The experimental settings are $(x_1, x_2, \tau) = (145^{\circ}C, 250^{\circ}C, 262.5)$.

Biographies

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