An Improved Procedure for Retrospectively Dating the Emergence and Collapse of Bubbles*

Mohitosh Kejriwal[†] *Purdue University*

Linh Nguyen[‡] *Purdue University*

Pierre Perron[§] *Boston University*

December 5, 2024

Abstract

This paper proposes a new ordinary least squares (OLS)-based procedure for retrospectively dating the emergence and collapse of bubbles. We first consider a data generating process that entails a switch from a unit root regime to an explosive regime followed by a collapse and subsequent return to unit root behavior. We demonstrate analytically that the standard OLS estimates are inconsistent and date both the origination and implosion points with a delay in large samples. A simple modification that involves omitting the residual corresponding to the implosion date is shown to yield consistent estimates. We also develop an efficient dating algorithm that can accommodate a framework with multiple bubbles. The algorithm exploits the explicit form of the unit root restrictions to directly embed them into the recursive optimization problem which obviates the need to rely on an iterative scheme that requires initial values. Extensive simulation experiments indicate that our proposed procedure typically delivers estimates with lower bias and root mean squared error relative to competing alternatives. An empirical illustration is included.

Keywords: bubbles, dynamic programming, dating algorithm, explosive, unit root, omission.

MSC Code: 62P20
JEL Code: C22

---

*We are grateful to David Harvey and Steve Leybourne (the Co-Editors), and an anonymous referee for constructive comments that helped improved the paper. Any errors are our own.

[†]Daniels School of Business, Purdue University, 403 West State Street, West Lafayette IN 47906 (mkejriwa@purdue.edu).

[‡]Daniels School of Business, Purdue University, 403 West State Street, West Lafayette IN 47906 (nguye535@purdue.edu).

[§]Department of Economics, Boston University, 270 Bay State Road, Boston MA 02215 (perron@bu.edu).

1 Introduction

The twin problems of detecting and dating the origination and subsequent implosion of bubbles have garnered considerable attention in the econometrics literature over the past two decades. These issues are of immense practical importance as policymakers can effectively use information on the presence of bubbles to devise and implement specific policies in order to mitigate their potentially adverse consequences. Accordingly, a multitude of procedures has been developed for ex post detection and date-stamping of bubble episodes as well as real-time monitoring for the origination of bubbles. The procedures have been applied to a wide variety of applications including stock, real estate, commodity, and art markets, as well as prices of cryptocurrencies, thereby testifying to their empirical relevance. Given the volume of literature on this topic, we do not attempt an exhaustive survey and instead refer to Hu (2023) and Skrobotov (2023) for recent comprehensive reviews of the literature.

This paper studies the problem of retrospectively dating the inception and implosion of bubbles conditional on their detection. This is an important issue since, as noted by Harvey et al. (2017), effective date-stamping strategies can provide useful information regarding the type of economic and financial events that are typically associated with bubble-like phenomena and thereby caution policymakers to take appropriate action in case similar events are deemed to occur in the near future. A plethora of methods has been proposed to address the problem of ex-post date-stamping which vary according to the particular bubble model specification adopted as well as whether the break dates (i.e., the dates of inception and implosion) are estimated jointly or recursively/sequentially.

Phillips et al. (2011) [PWY henceforth] proposed a recursive procedure based on right-tailed unit root tests to detect the presence of explosive behavior as well as date-stamp the origination and termination of such behavior. Their date-stamping algorithm hinges on comparing the sequence of recursively computed Augmented Dickey-Fuller (ADF) statistics with their corresponding right-tailed critical values. Phillips and Yu (2009) established the consistency of PWY's dating estimators assuming that the data generating process (DGP) is characterized by a single bubble. Specifically, the origination of the bubble was modeled as a transition from a unit root process to a mildly explosive process while its termination was modeled as an instantaneous collapse to a new level at which unit root behavior resumes and continues until the end of the sample. These testing and date-stamping procedures were subsequently extended to a multiple bubbles framework by Phillips et al. (2015a) [PSY henceforth] and their asymptotic properties were derived by Phillips et al. (2015b).

Harvey et al. (2017) [HLS henceforth] suggested an alternative date-stamping approach that jointly estimates the origination and collapse dates in a single bubble model based on minimizing the sum of squared residuals in combination with a Bayesian Information Criterion (BIC) for model selection. Instead of an abrupt crash as in Phillips and Yu (2009), the collapse mechanism in HLS entails a transition from an explosive to a stationary regime followed by a reversion to unit root behavior. Phillips and Shi (2018) established the large sample validity of the PSY approach under alternative forms of bubble implosion including the transient collapse dynamics espoused by HLS. Monte Carlo simulations reported in HLS show that their proposed dating procedure outperforms the PSY procedure in finite samples.

A different date-stamping approach involves estimating the origination and collapse dates sequentially. In a single bubble framework, Pang et al. (2021) showed that the collapse date can be consistently estimated by minimizing the sum of squared residuals in a single break model; the origination date is then estimated using the subsample preceding the estimated date of collapse. Although the estimated origination date is inconsistent, the timing as a fraction of the sample size was shown to be consistent. Kurozumi and Skrobotov (2023) extended the analysis in Pang et al. (2021) to allow for unit root behavior following the collapse. They obtained results similar to Pang et al. (2021) regarding the origination and collapse dates and additionally established conditions under which the date of recovery (i.e., the switch to the unit root path) can be consistently estimated. Finally, Harvey et al. (2020) [HLW henceforth] proposed a two-step approach to date-stamping multiple bubbles that offers an improvement over PSY's recursive approach. The first step involves using the PSY procedure to identify date windows in which explosive behavior starts and ends. In the second step, the date estimates are obtained by applying a model-based BIC approach within each date window.

This paper proposes a new ordinary least squares (OLS)-based procedure to retrospectively date the emergence and collapse of bubbles by minimizing a modified sum of squared residuals. Adopting the same DGP as Phillips and Yu (2009), we first demonstrate analytically that the standard OLS dating estimators obtained by minimizing the sum of squared residuals are inconsistent and date both the origination and implosion points with a delay. In particular, the estimate of the origination date is shown to converge to the true implosion date while the implosion date estimate converges to a date in the post-implosion period determined by the level of trimming employed. A simple modification of the OLS procedure that involves omitting the residual corresponding to the implosion date is shown to yield consistent estimates of both the origination and collapse dates.

A second contribution of our paper is to develop an efficient date-stamping algorithm that can simultaneously estimate the origination and collapse dates in a framework with multiple bubbles. While a brute-force grid search procedure is computationally very costly with multiple bubbles, the proposed algorithm yields equivalent estimates but only requires computing time comparable to that for a single bubble model. Our algorithm is a modification of existing dynamic programming algorithms proposed by Bai and Perron (2003) [BP henceforth] and Perron and Qu (2006) [PQ henceforth] for estimating a linear regression model with multiple breaks. In particular, our algorithm exploits the explicit form of the unit root restrictions (pertaining to the non-bubble regimes) to directly embed them into the recursive optimization problem which obviates the reliance on an iterative scheme that requires initial values. This feature alleviates our algorithm from the problem of local minima which can affect approaches based on iterative schemes. Extensive simulation experiments indicate that our proposed procedure typically delivers estimates with lower bias and root mean squared error relative to extant approaches. An application to oil prices illustrates the relevance of the proposed method in practice.

Our paper is closely related to earlier work by Kejriwal et al. (2013) who develop Wald tests of the unit root hypothesis against structural changes in persistence. Their model under the alternative hypothesis involves switches between unit root [$I(1)$] and stationary [$I(0)$] regimes without any discontinuities between regimes. They employ the iterative dynamic programming algorithm proposed by PQ to estimate the break dates subject to the unit root restrictions in the relevant regimes. The PQ algorithm extends the BP algorithm designed for unrestricted estimation of the break dates to allow for linear restrictions on the regression coefficients. In contrast to Kejriwal et al. (2013), our model entails discontinuities between regimes due to the abrupt implosion of the bubbles which necessitates a modification of the PQ algorithm (via omission of specific residuals) to ensure that the parameters are consistently estimated. Simulations show that our proposed algorithm often yields improved estimates relative to the modified version of the PQ algorithm. The source of this improvement emanates from the fact that the PQ algorithm relies on an iterative scheme that employs unrestricted break date estimates as initial values which can potentially inflate the variance of the final estimates in small samples.

The rest of the paper is organized as follows. Section 2 presents the basic model with a single bubble and derives the large sample properties of the standard and modified OLS estimators. Section 3 considers a general framework with multiple bubbles and develops an efficient algorithm for dating their emergence and collapse. Section 4 contains a set of Monte

Carlo experiments to assess the finite sample properties of our proposed estimators relative to existing alternatives. Section 5 presents an empirical illustration and section 6 concludes. The online supplement includes Appendices A and B which contain, respectively, the proofs of theoretical results and additional Monte Carlo results.

2 The Basic Model

We first consider a scalar random variable y_t generated by a single bubble specified as

$$y_t = \begin{cases} y_{t-1} + u_t, & 1 \leq t \leq T_1^0 \\ \delta y_{t-1} + u_t, & T_1^0 + 1 \leq t \leq T_2^0 \\ y_{T_1^0} + z^* + \sum_{j=T_2^0+1}^{t} u_j, & T_2^0 + 1 \leq t \leq T \end{cases} \tag{1}$$

where u_t is i.i.d. with $E(u_t) = 0$, $E(u_t^2) = \sigma^2$ and $y_0 = o_p(T^{1/2})$, $z^* = O_p(1)$. This is the same DGP adopted by Phillips and Yu (2009) and PWY: the stochastic process switches from an $I(1)$ regime to an explosive one at date $T_1^0 + 1$, followed by a collapse at date $T_2^0 + 1$ with a subsequent return to (pre-bubble) martingale behavior which continues until the end of the sample (T). We refer to (T_1^0, T_2^0) as the true break dates and $(\lambda_1^0, \lambda_2^0)$ as the true break fractions so that $T_1^0 = \lfloor \lambda_1^0 T \rfloor$ and $T_2^0 = \lfloor \lambda_2^0 T \rfloor$. For some small positive number ϵ, we define $\mathcal{T}_\epsilon(2) = \{(T_1, T_2);\ |T_2 - T_1| \geq \lfloor \epsilon T \rfloor, T_1 \geq \lfloor \epsilon T \rfloor, T_2 \leq \lfloor (1-\epsilon) T \rfloor\}$. The set $\mathcal{T}_\epsilon(2)$ contains candidate break dates (T_1, T_2) that are separated by a positive fraction ϵ (the level of trimming) of the sample size. Our objective is to consistently estimate the break dates (T_1^0, T_2^0) and the parameter δ that determines the degree of explosive behavior.

2.1 OLS Estimation

We will start with standard OLS estimation. The estimation procedure imposes the unit root restriction in the first and last regimes while estimating δ using an OLS regression of y_t on a constant and y_{t-1} using observations in the second regime as demarcated by the potential break dates (T_1, T_2). Define the following quantities:

$$\begin{aligned} \bar{y}_2 &= (T_2 - T_1)^{-1} \sum_{t=T_1+1}^{T_2} y_t, \quad \bar{y}_{2,-1} = (T_2 - T_1)^{-1} \sum_{t=T_1+1}^{T_2} y_{t-1}, \\ \hat{\delta}(T_1, T_2) &= \left[\sum_{t=T_1+1}^{T_2} (y_{t-1} - \bar{y}_{2,-1})^2 \right]^{-1} \sum_{t=T_1+1}^{T_2} (y_{t-1} - \bar{y}_{2,-1}) y_t. \end{aligned} \tag{2}$$

The OLS estimates of the parameters are obtained as

$$(\hat{T}_1, \hat{T}_2) = \arg \min_{(T_1, T_2) \in \mathcal{T}_\epsilon(2)} SSR(T_1, T_2),$$

where

$$SSR(T_1,T_2)=\sum_{t=2}^{T_1}(\Delta y_t)^2+\sum_{t=T_1+1}^{T_2}[y_t-\bar{y}_2-\hat{\delta}(T_1,T_2)(y_{t-1}-\bar{y}_{2,-1})]^2+\sum_{t=T_2+1}^{T}(\Delta y_t)^2 \quad (3)$$

is the sum of squared residuals based on candidate break dates (T_1, T_2). The estimate of δ is then obtained as $\hat{\delta}=\hat{\delta}(\hat{T}_1,\hat{T}_2)$.

Following HLS, our theoretical analysis models the autoregressive parameter δ as fixed and independent of the sample size. An alternative "mildly explosive" framework, developed by Phillips and Magdalinos (2007), models the parameter as being dependent on the sample size such that it converges to one at a slower rate than the sample size. The advantage of the latter framework is that it permits the application of an invariance principle that facilitates asymptotically pivotal inference. Since our interest lies in investigating the consistency/inconsistency properties of different estimators, we adopt the fixed parameter framework for our asymptotic analysis.

The large sample behavior of the OLS estimates is stated in the following result.

Theorem 1 *Suppose that y_t is generated by* (1) *with* $(T_1^0, T_2^0) \in \mathcal{T}_\epsilon(2)$. *Then we have*

(a) $\hat{T}_1 - T_2^0 \xrightarrow{p} 0,\ \hat{T}_2-(T_2^0+\lfloor \epsilon T \rfloor) \xrightarrow{p} 0;$

(b) $\hat{\delta} \xrightarrow{p} 0.$

Theorem 1 shows that the OLS estimates of the break dates are inconsistent with each break date estimate selecting a break date later than the corresponding true break date. Specifically, the second true break date T_2^0 is in fact consistently estimated by the first break date estimate $\hat{T}_1$ while the estimate $\hat{T}_2$ dates the termination of explosive behavior with a delay determined by the trimming level ϵ. Moreover, the OLS estimate of the autoregressive coefficient is also inconsistent and biased towards zero.

The intuition for this result can be understood as follows. First, there are four principal sources of contamination that may potentially affect $SSR(T_1, T_2)$. The first involves the squared difference between the first post-crash observation $y_{T_2^0+1}$ and the final observation in the explosive regime $y_{T_2^0}$. Any combination of (T_1, T_2) with $T_1 > T_2^0$ or $T_2 \leq T_2^0$ is affected by this form of contamination. The second source which is relevant when $T_1 < T_2^0$, $T_2 > T_2^0$ arises from the inclusion of both explosive and post-crash $I(1)$ observations when estimating δ which generates a mean-reverting behavior and hence imparts a downward bias to the autoregressive estimate. The third source emanates from incorrectly treating observations from the explosive regime as $I(1)$ observations and thus taking their first difference.

Any combination of (T_1, T_2) with $T_1 > T_1^0$ is affected by this form of contamination. The fourth and final source of contamination occurs when $T_1 = T_2^0,\ T_2 \geq T_2^0 + \lfloor \epsilon T \rfloor$ in which case δ is estimated using post-crash $I(1)$ observations in time periods $\{T_2^0+1, ..., T_2^0+\lfloor \epsilon T \rfloor\}$. Moreover, the extent of this contamination (in terms of its impact on SSR) increases with the number of post-crash observations included in this estimation sample.

Next, among the aforementioned sources of contamination, the first source is dominant, followed by the second, third and fourth sources, in that particular order. The reason is that the first source involves the first difference between an $I(1)$ and an explosive observation which entails a larger increase in the sum of squared residuals relative to the three other sources. The second source of contamination dominates the third since the former treats the explosive regime as stationary in large samples while the latter treats the explosive regime as $I(1)$. Finally, the fourth source is dominated by the others since it only involves the post-crash $I(1)$ observations while the others also involve explosive observations.

Combining the above facts, it follows that the sum of squared residuals is minimized at $\hat{T}_1 = T_2^0,\ \hat{T}_2 = T_2^0 + \lfloor \epsilon T \rfloor$. Theorem 1(b) follows from the fact that, in large samples, δ is estimated using observations in time periods $\{T_2^0 + 1, ..., T_2^0 + \lfloor \epsilon T \rfloor\}$. The sum of squared residuals from this estimation sample is minimized at $\hat{\delta} = 0$ to ensure that the effect of the explosive observation $y_{T_2^0}$ is asymptotically negligible.

2.2 Modified Estimation

To address the issue of inconsistency, we suggest a simple modification of the OLS procedure. In particular, we propose omitting the residual that corresponds to the potential collapse date $T_2 + 1$ when constructing the overall global sum of squared residuals. Thus, the modified OLS estimates are obtained as $(\widetilde{T}_1, \widetilde{T}_2) = \arg\min_{(T_1,T_2)\in\mathcal{T}_\epsilon(2)} SSR_{om}(T_1, T_2)$ and $\widetilde{\delta} = \hat{\delta}(\widetilde{T}_1, \widetilde{T}_2)$, where

$$SSR_{om}(T_1, T_2) = \sum_{t=2}^{T_1} (\Delta y_t)^2 + \sum_{t=T_1+1}^{T_2} [y_t - \bar{y}_2 - \hat{\delta}(T_1, T_2)(y_{t-1} - \bar{y}_{2,-1})]^2 + \sum_{t=T_2+2}^{T} (\Delta y_t)^2 \quad (4)$$

and "om" denotes omission. Note that unlike $SSR(T_1, T_2),\ SSR_{om}(T_1, T_2)$ omits the term $(\Delta y_{T_2+1})^2$. We label by $(\widetilde{T}_1, \widetilde{T}_2, \widetilde{\delta})$ and $(\hat{T}_1, \hat{T}_2, \hat{\delta})$ the estimators with and without omission, respectively. The consistency of the OLS estimates with omission is established in the following result.

Theorem 2 *Suppose that y_t is generated by* (1) *with $(T_1^0, T_2^0) \in \mathcal{T}_\epsilon(2)$. Then we have* *(a)* $\widetilde{T}_1 - T_1^0 \xrightarrow{p} 0,\ \widetilde{T}_2 - T_2^0 \xrightarrow{p} 0;$

(b) $\widetilde{\delta} \xrightarrow{p} \delta.$

The intuition for this result can be understood by referencing the four sources of contamination discussed above. First, note that none of these sources of contamination affect $SSR_{om}(T_1^0, T_2^0)$. The case $\widetilde{T}_2 < T_2^0$ is then ruled out because if $\widetilde{T}_2 < T_2^0$, $SSR_{om}(\widetilde{T}_1, \widetilde{T}_2)$ is affected by the first source of contamination. Similarly, the case $\widetilde{T}_2 > T_2^0$ is eliminated since if $\widetilde{T}_2 > T_2^0$, $SSR_{om}(\widetilde{T}_1, \widetilde{T}_2)$ would be impacted by the second and fourth sources of contamination. The case $\widetilde{T}_1 > T_1^0$ is ruled out since $SSR_{om}(\widetilde{T}_1, \widetilde{T}_2)$ would then be susceptible to the third source of contamination. Finally, the case $\widetilde{T}_1 < T_1^0$ is eliminated by the fact that $SSR_{om}(\widetilde{T}_1, \widetilde{T}_2)$ would then be affected by the contamination stemming from treating the pre-bubble $I(1)$ observations as observations from the explosive regime. Consequently, the sum of squared residuals is minimized in large samples when $\widetilde{T}_1 = T_1^0$, $\widetilde{T}_2 = T_2^0$. This discussion also suggests that the collapse date is likely to be more accurately estimated than the origination of explosiveness since any deviation of $\widetilde{T}_2$ from T_2^0 entails a larger increase in the sum of squared residuals (in terms of order of magnitude) than a similar deviation of $\widetilde{T}_1$ from T_1^0. This feature will be borne out in the simulations presented in Section 4.

HLS propose jointly estimating the origination and termination dates in a single bubble model by minimizing the sum of squared residuals. Instead of an instantaneous collapse as in (1), their collapse mechanism is modeled as a transition from an explosive to a stationary regime before the resumption of unit root behavior. The persistence of the stationary regime reflects the rate of adjustment following the termination of the bubble. A simple modification of the HLS procedure that incorporates an instantaneous collapse can be shown to yield break date estimates that are numerically identical to the estimates with omission $(\widetilde{T}_1, \widetilde{T}_2)$.[1] Specifically, consider estimating the bubble start and end dates by minimizing the sum of squared residuals from the regression

$$\Delta y_t = \hat{c}^*(T_1, T_2) D_t(T_1, T_2) + \hat{\delta}_1^*(T_1, T_2) D_t(T_1, T_2) y_{t-1} + \hat{\delta}_2^*(T_1, T_2) D_t(T_2, T_2+1) y_{t-1} + \hat{e}_t, \quad (5)$$

where $D_t(a, b) = 1(a < t \leq b)$, and $\{\hat{c}^*(T_1, T_2), \hat{\delta}_1^*(T_1, T_2), \hat{\delta}_2^*(T_1, T_2)\}$ are the fitted OLS estimates. Equation (5) modifies the regression equation for model 4 in HLS by replacing the dummy variable $D_t(T_2, T_3)$ (with T_3 denoting the date at which the process transitions to a stationary regime) with the one-time dummy variable $D_t(T_2, T_2 + 1)$. We will show that

[1] We thank the Co-Editors for pointing this out and helping us draw a connection between our proposed approach and that of HLS.

the sum of squared residuals from (5) is equivalent to $SSR_{om}(T_1, T_2)$ from (4) which yields the equivalence between the two sets of date estimates. To see this, observe that the sum of squared residuals from (5) is

$$SSR^*(T_1, T_2) = \sum_{t=2}^{T_1} (\Delta y_t)^2 + \sum_{t=T_1+1}^{T_2} [\Delta y_t - \hat{\delta}_1^*(T_1, T_2) y_{t-1} - \hat{c}^*(T_1, T_2)]^2 + \hat{e}_{T_2+1}^2 + \sum_{t=T_2+2}^{T} (\Delta y_t)^2, \tag{6}$$

where

$$\hat{e}_{T_2+1}^2 = \left\{ \Delta y_{T_2+1} - \hat{\delta}_2^*(T_1, T_2) y_{T_2} \right\}^2 = \left\{ \Delta y_{T_2+1} - y_{T_2}^{-2}(y_{T_2} \Delta y_{T_2+1}) y_{T_2} \right\}^2 = 0. \tag{7}$$

Further, we have

$$\hat{\delta}_1^*(T_1, T_2) = \left[\sum_{t=T_1+1}^{T_2} (y_{t-1} - \bar{y}_{2,-1})^2 \right]^{-1} \sum_{t=T_1+1}^{T_2} (y_{t-1} - \bar{y}_{2,-1}) \Delta y_t = \hat{\delta}(T_1, T_2) - 1, \tag{8}$$

and

$$\begin{aligned} \hat{c}^*(T_1, T_2) &= \bar{y}_2 - \bar{y}_{2,-1} - \hat{\delta}_1^*(T_1, T_2) \bar{y}_{2,-1} = \bar{y}_2 - \bar{y}_{2,-1} - \{ \hat{\delta}(T_1, T_2) - 1 \} \bar{y}_{2,-1} \\ &= \bar{y}_2 - \hat{\delta}(T_1, T_2) \bar{y}_{2,-1} = \hat{c}(T_1, T_2), \end{aligned} \tag{9}$$

where $\hat{\delta}(T_1, T_2)$ is defined in (2) and $\hat{c}(T_1, T_2)$ is the intercept estimate from (4). Substituting (7)-(9) in (6), the equivalence between $SSR^*(T_1, T_2)$ and $SSR_{om}(T_1, T_2)$ follows.

3 The General Model with Multiple Bubbles

This section proposes a new algorithm for estimating multiple break dates that can improve upon existing approaches. To this end, we consider a generalization of model (1) that can accommodate multiple bubbles:

$$\begin{aligned} y_t &= (\delta_i y_{t-1} + u_t) \mathbf{1}(\delta_i > 1) + \left(\sum_{s=T_{i-1}^0+1}^{T_i^0} u_s + y_{T_{i-1}^0}^* \right) \mathbf{1}(\delta_i = 1), \\ y_{T_{i-1}^0}^* &= y_{T_{i-2}^0} \mathbf{1}(i > 1) + z_i^*; \quad z_i^* = O_p(1), \end{aligned} \tag{10}$$

where $1 \leq i \leq m+1$ with the convention $T_0^0 = 0$ and $T_{m+1}^0 = T$. The process is therefore subject to m breaks or $m+1$ regimes with break dates $(T_1^0, ..., T_m^0)$. When m is even, there are $m/2$ or $(m/2+1)$ explosive regimes when the initial regime has a unit root $[I(1)]$ or is explosive, respectively. When m is odd, there are $(m+1)/2$ regimes of explosive behavior regardless of whether the initial regime is $I(1)$ or explosive. In this paper, we will primarily consider the case where the initial regime is $I(1)$ and briefly discuss the case with an initial explosive regime (see Remark 1 below).

Given the inconsistency of the standard OLS estimators as demonstrated in Section 2, we focus on modified least squares estimation that involves omitting the residuals corresponding to the potential bubble implosion dates. The modified estimates of the break dates are obtained as $(\widetilde{T}_1, ..., \widetilde{T}_m) = \arg\min_{(T_1,...,T_m)\in\mathcal{T}_\epsilon(m)} SSR_{om}(T_1, ..., T_m)$ where $\mathcal{T}_\epsilon(m) = \{(T_1, ..., T_m);\ |T_{i+1} - T_i| \geq \lfloor \epsilon T \rfloor, T_1 \geq \lfloor \epsilon T \rfloor, T_m \leq \lfloor (1-\epsilon)T \rfloor\}$ and

$$\begin{aligned} SSR_{om}(T_1, ..., T_m) &= \sum_{t=2}^{T_1}(\Delta y_t)^2 + \sum_{t=T_1+1}^{T_2}[y_t - \hat{\delta}_2(T_1,T_2)y_{t-1} - \hat{c}_2(T_1,T_2)]^2 + \sum_{t=T_2+2}^{T_3}(\Delta y_t)^2 \\ &+ ... + \sum_{t=T_m+1+[1-l(m)]}^{T}[y_t - \{l(m)\hat{\delta}_{m+1}(T_m,T_{m+1}) + (1-l(m))\}y_{t-1} \\ &-l(m)\hat{c}_{m+1}(T_m,T_{m+1})]^2, \qquad (11) \end{aligned}$$

with $l(m) = 1$ if m is odd, and zero otherwise. The estimates $(\hat{c}_i(T_{i-1}, T_i), \hat{\delta}_i(T_{i-1}, T_i))$ are obtained from an OLS regression of y_t on a constant and y_{t-1} using observations $t = T_{i-1}+1, T_{i-1}+2, ..., T_i$. A standard grid search procedure to minimize (11) would require least squares operations of order $O(T^m)$ and thus be computationally very expensive for $m > 2$. An efficient approach to this problem is to employ the principle of dynamic programming that only requires operations of order $O(T^2)$ regardless of the number of breaks. BP and PQ develop algorithms based on this principle for estimating multiple breaks in a linear regression model. In order to motivate our proposed algorithm, we first discuss in Section 3.1 the PQ algorithm which is an extension of the BP algorithm. The proposed algorithm is then presented in Section 3.2.

3.1 The Perron and Qu (2006) Algorithm

PQ propose a computationally efficient dynamic programming algorithm to estimate the break dates in a linear regression framework with multiple breaks subject to a set of linear restrictions on the regression coefficients. Their algorithm extends the BP algorithm designed for unrestricted estimation of the break dates in order to obtain more precise (i.e., lower variance) estimates. Kejriwal et al. (2013) use the PQ algorithm to estimate the break dates in an autoregressive model characterized by switches between $I(1)$ and $I(0)$ regimes where the $I(1)$ restrictions are imposed in the relevant regimes. In contrast to Kejriwal et al. (2013), our model involves discontinuities between regimes due to abrupt implosion of the bubbles that necessitates the omission of particular residuals to ensure consistent estimation.

The PQ algorithm does not omit the residuals for any of the time periods but can be easily modified to allow for such omission. We will henceforth refer to this modification as the PQ algorithm with omission. From (11), it is evident that our optimization problem is a special case of that considered in PQ which imposes the restrictions $c_i = 0, \delta_i = 1$ in the $I(1)$ regimes. Thus, the PQ algorithm with omission can be employed to obtain the break date estimates by recasting our problem within their framework.

The PQ algorithm entails the use of an iterative scheme that iterates between estimating the break dates and the regression coefficients until convergence. The initialization step in this scheme employs the unrestricted BP estimates. As with any iterative procedure, whether a global or local minimum is achieved depends on the initial values. In particular, the precision of the algorithm depends crucially on the first step estimates of the break dates. As noted by PQ (p. 383), in cases where the global minimum is not achieved, the estimates are typically very far from the true values, often at the beginning or the end of the sample.

3.2 The Proposed Algorithm

Motivated by the preceding discussion, we develop an efficient dating algorithm that exploits the explicit form of the $I(1)$ restrictions (i.e., the parameters in some regimes taking specific values) to directly incorporate them in the optimization problem, thereby obviating the reliance on initial values. Unlike the PQ estimates, our proposed estimates are equivalent to those obtained from a grid search procedure since the restricted sum of squared residuals can be computed directly without resorting to an iterative scheme. Monte Carlo simulations conducted in Section 4 demonstrate that our proposed algorithm often delivers estimates with improved statistical properties compared to the PQ algorithm (with or without omission). In particular, we will show that the PQ estimates often incur higher variance than our recommended estimates which stems from the relatively high variance of the BP estimates used as initial values.

To describe the proposed algorithm, we introduce the following notation. Let $SSR_1(1, j) = \sum_{t=2}^{j}(\Delta y_t)^2$. For $i = 2, ..., m+1$, let

$$SSR_i(j+1, n) = \begin{cases} \sum_{t=j+1}^{n}(y_t - \hat{\delta}_i y_{t-1} - \hat{c}_i)^2, & \text{if } i \text{ is even} \\ \sum_{t=j+2}^{n}(\Delta y_t)^2, & \text{if } i \text{ is odd} \end{cases}$$

The implementation of the proposed algorithm involves the following steps:

1. Compute and store the triangular matrix of the global unrestricted sums of squared residuals $GSSR^u$, the triangular matrix of the global restricted sums of squared residuals with omission $GSSR^r_{om}$, and a vector $VSSR_1$ containing all permissible sums of squared residuals $SSR_1(1,j)$ (details of this step are provided below).

2. Compute and store the restricted sums of squared residuals $SSR_{om}(\{T_{1,n}\})$, for $2h \leq n \leq T-(m-1)h$, where $h = \lfloor \epsilon T \rfloor$, by solving the following dynamic programming problem:
$$SSR_{om}(\{T_{1,n}\}) = \min_{h \leq j \leq n-h} [SSR_1(1,j) + SSR_2(j+1,n)].$$

3. Sequentially compute and store $SSR(\{T_{r,n}\})$ for $r = 2, ..., m-1$, with n ranging from $(r+1)h$ to $T-(m-r)h$. This is achieved by solving the following problem:
$$SSR_{om}(\{T_{r,n}\}) = \min_{rh \leq j \leq n-h} [SSR_{om}(\{T_{r-1,j}\}) + SSR_{r+1}(j+1,n)],$$
where $SSR_i(j+1,n)$ is the entry $(j+1,n)$ of $GSSR^u$ if i is even, or the entry $(j+1,n)$ of $GSSR^r_{om}$ if i is odd.

4. Finally, compute
$$SSR_{om}(\{T_{m,T}\}) = \min_{mh \leq j \leq T-h} [SSR_{om}(\{T_{m-1,j}\}) + SSR_{m+1}(j+1,T)].$$

Remark 1 *The algorithm is easily modified to accommodate the case in which the starting regime is explosive instead of $I(1)$. In fact, both cases can be nested within a general framework at the expense of some additional notation. Define the following quantities:*
$$SSR_1(1,j,\theta_1) = \begin{cases} \sum_{t=2}^{j}(y_t - \hat{\delta}_1 y_{t-1} - \hat{c}_1)^2, & \text{if } \theta_1 = 1, \\ \sum_{t=2}^{j}(\Delta y_t)^2, & \text{if } \theta_1 = 0. \end{cases}$$
And, for $i = 2, ..., m+1$,
$$SSR_i(j+1,n,\theta_i) = \begin{cases} \sum_{t=j+1}^{n}(y_t - \hat{\delta}_i y_{t-1} - \hat{c}_i)^2, & \text{if } \theta_i = 1, \\ \sum_{t=j+2}^{n}(\Delta y_t)^2, & \text{if } \theta_i = 0. \end{cases}$$
For an $I(1)$ starting regime, we set $\theta_i = 0$ if i is odd and $\theta_i = 1$, otherwise. Similarly, for an explosive starting regime, we set $\theta_i = 1$ if i is odd, and $\theta_i = 0$, otherwise. Then, we only need to replace $SSR_1(1,j)$ by $SSR_1(1,j,\theta_1)$ and $SSR_2(j+1,n)$ by $SSR_2(j+1,n,\theta_2)$ in step 2 above, $SSR_{r+1}(j+1,n)$ by $SSR_{r+1}(j+1,n,\theta_{r+1})$ in step 3, and $SSR_{m+1}(j+1,T)$ by $SSR_{m+1}(j+1,T,\theta_{m+1})$ in step 4.

We now discuss the computation of the quantities $GSSR^u$, $GSSR^r_{om}$, and $VSSR_1$ involved in step 1 of the algorithm. The dynamic programming algorithm of BP uses the triangular matrix of unrestricted sums of squared residuals (SSRs) for all permissible segments to search for the optimal break dates. Specifically, the algorithm requires storage of these SSRs that would be considered when searching over the optimal partition of break dates. Let this triangular matrix of global unrestricted SSRs be denoted $GSSR^u$. The unrestricted SSR for an (i, j) segment (with starting date i and ending date j), denoted $SSR^u_{i,j}$, is stored in entry (i, j) of $GSSR^u$. Similarly, we can store the restricted SSRs for all permissible (i, j) segments, obtained by imposing a unit root and omitting the first observation of the segment, in a different triangular matrix, say, $GSSR^r_{om}$. The entry (i, j) of $GSSR^r_{om}$ is denoted $SSR^{r,om}_{i,j}$. The quantities $SSR^u_{i,j}$ and $SSR^{r,om}_{i,j}$ are computed recursively via

$$\begin{aligned} SSR^u_{i,j} &= SSR^u_{i,j-1} + \hat{u}^2_{i,j}, \\ SSR^{r,om}_{i,j} &= SSR^{r,om}_{i,j-1} + (\Delta y_j)^2, \quad SSR^{r,om}_{i,i+1} = 0, \end{aligned}$$

with the recursive residuals $\hat{u}_{i,j}$ obtained as [see Brown et al. (1975)]

$$\hat{u}_{i,j} = \frac{y_j - y_{j-1}\hat{\delta}_{[i:j-1]} - \hat{c}_{[i:j-1]}}{\sqrt{1 + \boldsymbol{y}'_{j-1}\left(\boldsymbol{y}'_{[i:j-2]}\boldsymbol{y}_{[i:j-2]}\right)^{-1}\boldsymbol{y}_{j-1}}},$$

with $\boldsymbol{y}_j = (1, y_j)'$, $\boldsymbol{y}_{i:j} = (\boldsymbol{y}_i, \boldsymbol{y}_{i+1}, ..., \boldsymbol{y}_j)'$, where $\hat{\delta}_{[a:b]}$ and $\hat{c}_{[a:b]}$ are the estimates obtained using observations from a to b. Finally, the elements of the vector $VSSR_1$ are computed as $VSSR_1(j) = SSR_1(1, j)$ for $h \leq j \leq T - mh$, where

$$\begin{aligned} SSR_1(1, j) &= SSR_1(1, j-1) + (\Delta y_j)^2; \quad h+1 \leq j \leq T - mh \\ SSR_1(1, h) &= \sum_{t=2}^{h} (\Delta y_t)^2. \end{aligned}$$

In what follows, we will refer to our proposed algorithm as the joint search (JS) algorithm with omission. To examine the impact of omission, the Monte Carlo analysis in Section 4 also includes a version of the JS algorithm without omission. This version can be implemented following similar steps as those above except that no residuals are omitted at any step.

4 Monte Carlo Evidence

This section presents a set of Monte Carlo simulations to numerically evaluate the finite sample performance of the different dating estimators as well as to assess the adequacy of the large sample results derived in section 2. The simulation design is based on the DGP specified in (10) with $m \in \{2,4\}$, i.e., one and two bubbles. The details for each DGP are as follows.

DGP-1: A single bubble model specified by (1) with $(\lambda_1^0, \lambda_2^0) \in \{(0.5, 0.65), (0.4, 0.6)\}$ and $\delta \in \{1.02, 1.05\}$.

DGP-2: A two bubbles model specified by

$$y_t = \begin{cases} y_{t-1} + u_t, & 1 \leq t \leq T_1^0, \\ \delta_1 y_{t-1} + u_t, & T_1^0 + 1 \leq t \leq T_2^0, \\ y_{T_1^0} + z_1^* + \sum_{j=T_2^0+1}^{t} u_j, & T_2^0 + 1 \leq t \leq T_3^0, \\ \delta_2 y_{t-1} + u_t, & T_3^0 + 1 \leq t \leq T_4^0, \\ y_{T_3^0} + z_2^* + \sum_{j=T_4^0+1}^{t} u_j & T_4^0 + 1 \leq t \leq T. \end{cases}$$

For the break locations, we follow the design used in Table 7 of PSY, which corresponds to $\lambda_1^0 = 0.2$, $\lambda_2^0 \in \{0.3, 0.35, 0.4\}$, $\lambda_3^0 = 0.6$ and $\lambda_4^0 \in \{0.7, 0.75, 0.8\}$, yielding nine possible combinations for the duration of the first ($\lambda_2^0 - \lambda_1^0 \in \{0.1, 0.15, 0.2\}$) and second bubble ($\lambda_4^0 - \lambda_3^0 \in \{0.1, 0.15, 0.2\}$). For brevity, we only report the results for the cases with $\lambda_i^0 - \lambda_{i-1}^0 = 0.2$; $i = 2, 4$. The results for the other cases are qualitatively similar and available upon request. The autoregressive parameters are set to $\delta_1 = \delta_2 = 1.05$.

We experiment with three different serial correlation structures in the error component $\{u_t\}$. In the first case, $u_t \sim i.i.d.\ \mathcal{N}(0,1)$. In the second, $\{u_t\}$ follows an AR(1) process: $u_t = 0.5u_{t-1} + e_t$, $e_t \sim i.i.d.\ \mathcal{N}(0,1)$. Finally, the third case considered is a MA(1) structure for $\{u_t\}$: $u_t = e_t + 0.5e_{t-1}$, $e_t \sim i.i.d.\ \mathcal{N}(0,1)$. To save space, we only present in the main text the results for the i.i.d. case and defer the other cases to Appendix B.

In all experiments, the perturbations z^*, z_1^*, z_2^* are randomly drawn from a $\mathcal{N}(1,1)$ distribution. The level of trimming is set to $\epsilon \in \{0.05, 0.10\}$. We only report results for $\epsilon = 0.10$ since those for $\epsilon = 0.05$ were very similar. The number of replications is 5,000.

The rest of this section is organized as follows. Section 4.1 discusses the alternative dating estimators used in the Monte Carlo comparison; Sections 4.2 and 4.3 report results for the single bubble case (DGP-1) and the two bubbles case (DGP-2), respectively; Section 4.4 illustrates that the JS procedure often yields a smaller sum of squared residuals than the PQ procedure, suggesting that the latter may only attain a local minimum; Section 4.5 discusses the issue of selecting the number of bubbles and its impact on the performance of the dating estimators.

4.1 Alternative Dating Estimators

Our simulation design includes a comparison of six dating estimators: the PQ estimators with and without omission, the JS estimators with and without omission, the PSY estimator, and the HLW estimator. Throughout, we use "^" to denote an estimator "without omission" and "~" to denote an estimator "with omission". We use the single bubble case ($m = 2$) to illustrate the first five estimators and the two bubbles case ($m = 4$) to illustrate the HLW estimator given that it is applicable to multiple bubbles.

1. *PQ estimator without omission.* This estimator, denoted $(\hat{T}_1^{PQ}, \hat{T}_2^{PQ})$, solves the minimization problem

$$\underset{(T_1,T_2)}{\arg\min} \left\{ \sum_{t=2}^{T_1} (\Delta y_t)^2 + \sum_{t=T_1+1}^{T_2} [y_t - \hat{\delta}(T_1,T_2) y_{t-1} - \hat{c}_2(T_1,T_2)]^2 + \sum_{t=T_2+1}^{T} (\Delta y_t)^2 \right\}, \quad (12)$$

using as initial values the unrestricted BP estimates obtained by solving the minimization problem

$$\underset{(T_1,T_2)}{\arg\min} \left\{ \begin{array}{c} \sum_{t=2}^{T_1} \left[y_t - \hat{\delta}_1(T_1,T_2) y_{t-1} - \hat{c}_1(T_1,T_2) \right]^2 \\ + \sum_{t=T_1+1}^{T_2} \left[y_t - \hat{\delta}_2(T_1,T_2) y_{t-1} - \hat{c}_2(T_1,T_2) \right]^2 \\ + \sum_{t=T_2+1}^{T} \left[y_t - \hat{\delta}_3(T_1,T_2) y_{t-1} - \hat{c}_3(T_1,T_2) \right]^2 \end{array} \right\},$$

where $\left\{ \hat{c}_i(T_1,T_2), \hat{\delta}_i(T_1,T_2) \right\}$ are the unrestricted parameter estimates for regime i based on the partition (T_1, T_2).

2. *JS estimator without omission.* This estimator, denoted $(\hat{T}_1^{JS}, \hat{T}_2^{JS})$, solves the same minimization problem as (12) but uses the version of the dynamic programming algorithm proposed in section 3 that does not involve omission.

3. *PQ estimator with omission.* This estimator, denoted $(\widetilde{T}_1^{PQ}, \widetilde{T}_2^{PQ})$, solves the minimization problem

$$\underset{(T_1,T_2)}{\arg\min} \left\{ \sum_{t=3}^{T_1} (\Delta y_t)^2 + \sum_{t=T_1+2}^{T_2} [y_t - \widetilde{\delta}(T_1,T_2) y_{t-1} - \widetilde{c}_2(T_1,T_2)]^2 + \sum_{t=T_2+2}^{T} (\Delta y_t)^2 \right\}, \quad (13)$$

using as initial values the unrestricted BP estimates with omission obtained by solving the minimization problem

$$\arg\min_{(T_1,T_2)} \left\{ \begin{array}{c} \sum_{t=3}^{T_1} \left[y_t - \widetilde{\delta}_1(T_1,T_2)y_{t-1} - \widetilde{c}_1(T_1,T_2) \right]^2 \\ + \sum_{t=T_1+2}^{T_2} \left[y_t - \widetilde{\delta}_2(T_1,T_2)y_{t-1} - \widetilde{c}_2(T_1,T_2) \right]^2 \\ + \sum_{t=T_2+2}^{T} \left[y_t - \widetilde{\delta}_3(T_1,T_2)y_{t-1} - \widetilde{c}_3(T_1,T_2) \right]^2 \end{array} \right\},$$

where $\left\{ \widetilde{c}_i(T_1,T_2), \widetilde{\delta}_i(T_1,T_2) \right\}$ are the unrestricted parameter estimates for regime i based on the partition (T_1,T_2) that are obtained by omitting the first observation of the regime.

4. *JS estimator with omission.* This estimator, denoted $(\widetilde{T}_1^{JS}, \widetilde{T}_2^{JS})$, solves the same minimization problem as (13) but uses the dynamic programming algorithm proposed in section 3 with omission.

5. *PSY estimator.* The PSY estimator is based on a test statistic that entails taking the supremum of recursively computed backward and forward Augmented Dickey-Fuller (ADF) statistics. Specifically, this estimator, denoted $(\hat{T}_1^{PSY}, \hat{T}_2^{PSY})$, is obtained as

$$\hat{T}_1^{PSY} = \lfloor T\hat{r}_e \rfloor - 1; \quad \hat{r}_e = \inf_{r_2 \in [r_0,1]} \left\{ BSADF_{r_2}(r_0) > cv_{r_2}^{\alpha} \right\}, \tag{14}$$

$$\hat{T}_2^{PSY} = \lfloor T\hat{r}_f \rfloor - 1; \quad \hat{r}_f = \inf_{r_2 \in [\hat{r}_e + \ln(T)/T,1]} \left\{ BSADF_{r_2}(r_0) < cv_{r_2}^{\alpha} \right\}, \tag{15}$$

where $BSADF_{r_2}(r_0) = \sup_{r_1 \in [0, r_2 - r_0]} \left\{ ADF_{r_1}^{r_2} \right\}$, $ADF_{r_1}^{r_2}$ denotes the ADF statistic computed using the observations $\lfloor Tr_1 \rfloor + 1, ..., \lfloor Tr_2 \rfloor$, and $cv_{r_2}^{\alpha}$ is the $100(1-\alpha)\%$ critical value of the $BSADF_{r_2}(r_0)$ statistic based on $\lfloor Tr_2 \rfloor$ observations. We follow PSY in setting the minimum window width (as a fraction of the sample size) to $r_0 = 0.01 + 1.8T^{-1/2}$ and restricting the duration of bubble(s) to be at least $\lfloor \ln T \rfloor$ for implementing their test procedure. Following HLS, a lag length of one (i.e., one lag of Δy_t) is used to construct the ADF regressions. A nominal size of 5% is used and the finite sample critical values are simulated under the null hypothesis of a random walk with no drift and i.i.d. $\mathcal{N}(0,1)$ errors with 10,000 replications.[2]

6. *HLW estimator.* HLW propose a two-step dating procedure that extends the single bubble procedure developed by HLS to multiple bubbles. In the first step, the PSY

[2] We also simulated the critical values under the null hypothesis of a random walk with a small drift as in PSY. The results were very similar with no qualitative differences.

date estimates are used to split the sample into date windows each containing a single explosive episode. The second step entails the application of a BIC-based date estimation algorithm to obtain improved estimates. Since HLW (as HLS) consider a framework which models the collapse as gradual instead of abrupt, we adopt a modified version of their procedure that facilitates an abrupt collapse as detailed in Section 2.2. Moreover, since our proposed approach does not address model selection, we assume that each explosive episode follows DGP (1) instead of applying the BIC to each date window. This ensures a fair comparison of the procedures. Specifically, the following steps are involved in implementing the HLW procedure:

(a) Apply the PSY dating procedure to obtain two pairs of break fraction estimates $(\hat{r}_{1e}, \hat{r}_{1f})$ and $(\hat{r}_{2e}, \hat{r}_{2f})$, where $\hat{r}_{je}$ $(\hat{r}_{jf})$ is the estimated date of emergence (collapse) of the j-th bubble $(j = 1, 2)$.

(b) Partition the sample into two sub-sample date windows, denoted by (s_1, e_1) and (s_2, e_2), with $s_1 = 1$ and $e_2 = T$. Set $e_1 = \lfloor \hat{r}_{1f} T \rfloor + (\lfloor \hat{r}_{2e} T \rfloor - \lfloor \hat{r}_{1f} T \rfloor)/2$.

(c) Obtain the refined date estimates for the first bubble as $(\hat{T}_{11}, \hat{T}_{12})$ as

$$(\hat{T}_{11}, \hat{T}_{12}) = \underset{1<T_{11}<T_{12}<e_1}{\arg\min} \; SSR_w(T_{11}, T_{12}), \tag{16}$$

where $SSR_w(T_{11}, T_{12})$ is the sum of squared residuals from the regression over the first date window $[1, e_1]$:

$$\Delta y_t = \hat{c}_1 D_t(T_{11}, T_{12}) + \hat{\delta}_{11} D_t(T_{11}, T_{12}) + \hat{\delta}_{12} D_t(T_{12}, T_{12}+1) y_{t-1} + \hat{e}_t, \tag{17}$$

where $D_t(a, b) = 1(a < t \leq b)$.

(d) Given $(\hat{T}_{11}, \hat{T}_{12})$, set $s_2 = \hat{T}_{12}+1$. The refined date estimates for the second bubble are then obtained as

$$(\hat{T}_{21}, \hat{T}_{22}) = \underset{s_2<T_{21}<T_{22}<T}{\arg\min} \; SSR_w(T_{21}, T_{22}), \tag{18}$$

where $SSR_w(T_{21}, T_{22})$ is the sum of squared residuals from the regression over the second date window $[s_2, T]$:

$$\Delta y_t = \hat{c}_2 D_t(T_{21}, T_{22}) + \hat{\delta}_{21} D_t(T_{21}, T_{22}) + \hat{\delta}_{22} D_t(T_{22}, T_{22}+1) y_{t-1} + \hat{e}_t. \tag{19}$$

Following HLW, we impose the additional restrictions (minimum segment length requirements) $(\hat{T}_{j2} - \hat{T}_{j1})/T \geq 0.1$ and $\hat{T}_{j1}/T \geq 0.1$, for $j \in \{1, 2\}$.

All of the methods described above require a choice for the number of bubbles. PSY propose a procedure to estimate the number of bubbles based on repeated implementation of the crossing rules (14) and (15). Specifically, the estimated number of bubbles is the number of pairs ($\hat{r}_e$ $\hat{r}_f$) over the full sample that satisfy crossing rules of the form (14) and (15) with $\hat{r}_f - \hat{r}_e \geq \ln(T)/T$ (see PSY, p.1056 for further details). Since our paper presumes the presence of bubbles as opposed to testing for their presence, our simulation results are conditioned on 5,000 replications in which the PSY procedure identifies the same number of bubbles as present in the DGP. For example, if the DGP is characterized by two bubbles, we eliminate those replications in which the PSY procedure identifies a smaller/larger number of bubbles and increase the number of replications until we obtain 5,000 replications in which the procedure selects exactly two bubbles.[3] Section 4.5 discusses the implications of incorrectly selecting the number of bubbles.

4.2 Results for the Single Bubble Case

Table 1 presents results on the accuracy of the break date estimators in terms of how frequently they select the corresponding true break dates or select later dates. Specifically, we report the following for $j \in \{\text{JS, PQ}\}$: (i) $\hat{p}_i^C(j)$, which denotes the probability of "correctly" selecting the i-th break date using procedure j without omission; (ii) $\hat{p}_i^L(j)$, which denotes the probability of selecting the i-th break date "later" than the true break date using procedure j without omission; (iii) $\widetilde{p}_i^C(j)$, $\widetilde{p}_i^L(j)$ are defined similarly when procedure j is applied with omission. Finally, $\hat{p}_i^C(PSY)$ and $\hat{p}_i^L(PSY)$ denote, respectively, the probabilities of correctly selecting the i-th break date and selecting the i-th break date "later" than the true break date using the PSY procedure. The HLW results are not separately reported here since the HLS date estimates (i.e., the single bubble counterparts of the HLW estimates) based on the modified regression (5) are identical to the JS estimates with omission (see Section 2.2).

Consider first the estimates of the first break date corresponding to the origination of the bubble (Panel A). Regardless of the dating method employed, the true break date is selected only very infrequently, i.e., all methods are inadequate at dating the initiation of the bubble. In particular, the probability of selecting a date later than the true date is considerable for each of the methods and typically increases with the sample size.[4] Notwithstanding the

[3] We also considered an alternative design adopted by HLS where we condition our results on 5,000 replications in which the bubble(s) with the longest duration is (are) chosen for dating purposes if the PSY procedure identifies a larger number of bubbles than present in the DGP. The results were qualitatively similar.

[4] Kurozumi (2021), inter alia, studies the large sample properties of different bubble monitoring tests and

overall deficiency of the methods, the JS procedure with omission has the highest accuracy in dating the onset of explosive behavior relative to the other methods.

Turning to the estimates of the implosion date (Panel B), we note that the estimates with omission are quite effective in that they correctly identify the implosion date with high probability and their accuracy increases with the sample size in all cases. This feature is in accordance with the consistency result derived in Theorem 2. In contrast, the JS and PQ estimators without omission tend to select a date later than the crash date, and more so as the sample size increases, consistent with the prediction of Theorem 1. The JS estimator with omission again emerges as the preferred estimator as it uniformly dominates its competitors in dating the termination of explosive behavior.

The fact that the collapse of the bubble can be dated with much higher accuracy relative to its inception follows from the fact that the implosion embodies a much stronger signal than the origination of explosive behavior so that any deviation of the second break date from its true value leads to a larger increase in the sum of squared residuals than a comparable deviation of the first break date from its true counterpart.[5] Formally, as shown in the proof of Theorem 2, $\{SSR_{om}(T_1^0 + k_1, T_2^0) - SSR_{om}(T_1^0, T_2^0)\}$ for $k_1 \neq 0$ diverges at a slower rate than $\{SSR_{om}(T_1^0, T_2^0 + k_2) - SSR_{om}(T_1^0, T_2^0)\}$ for $k_2 \neq 0$.

Given the relatively low accuracy of the dating methods at estimating the origination of the bubble, Table 2 reports the bias and root mean squared error (RMSE) of the dating estimators when viewed as estimating the break fractions λ_1^0 and λ_2^0 instead of the break dates T_1^0 and T_2^0. The following findings are noteworthy. First, the JS/PQ estimators without omission are upward biased in all cases. Moreover, the magnitudes of the bias and RMSE of these estimators are consistent with the asymptotic theory presented in Section 2. For example, consider the JS estimator without omission. When $T = 400,\ \delta = 1.05,\ (\lambda_1^0, \lambda_2^0) = (0.4, 0.6)$, the bias and RMSE of the first break date estimator are both around 0.2, while the bias and RMSE of the second break date estimator are both around 0.10. Since $\epsilon = 0.10$, these values are in compliance with our large sample result that $\hat{\lambda}_1 \xrightarrow{p} \lambda_2^0,\ \hat{\lambda}_2 \xrightarrow{p} \lambda_2^0 + \epsilon$. Second, the absolute bias of the JS/PQ estimators with omission as well as the PSY estimator decline monotonically as the sample size increases in most cases and in all cases when considering the RMSE. Furthermore, given the sample size and the break locations,

finds that they tend to detect bubbles with a delay, and shows that their relative performance depends on whether the bubble emerges early or late in the monitoring period.

[5] In a similar vein, Pang et al. (2021) and Kurozumi and Skrobotov (2023) adopt a sample splitting approach where the breaks are estimated one at a time and find that the collapse date is typically estimated with higher accuracy than the origination date.

the performance of these estimators in terms of both bias and RMSE improves as the degree of explosiveness (δ) increases. Similarly, a longer duration of the explosive regime holding the other parameters fixed also induces an improvement in performance. Third, in terms of bias, while the JS estimator with omission typically dominates the other estimators when $T = 400$, the PQ estimator with omission is preferred with smaller sample sizes. Fourth, the JS procedure with omission delivers the smallest RMSE in the majority of cases.

Finally, Table 3 reports the bias and RMSE of the autoregressive coefficient estimators when evaluated at the estimated break dates. The main findings are as follows. First, the estimators are downward biased regardless of the procedure employed, the sample size, and the parameter values. Second, the magnitude of the biases incurred by the JS/PQ estimators without omission are substantial and increase with the sample size. Third, the RMSE of the JS/PQ estimators without omission either increase with the sample size (for the PQ procedure) or share a non-monotonic relationship with the sample size (for the JS procedure). In addition, the bias/RMSE magnitudes are again consistent with our theoretical result that $\hat{\delta} \xrightarrow{p} 0$ without omission: for example, when $T = 400$, $\delta = 1.05$, $(\lambda_1^0, \lambda_2^0) = (0.4, 0.6)$, $\left|\text{bias}(\hat{\delta})\right| \simeq \text{RMSE}(\hat{\delta}) \simeq 1.045$. Fourth, the bias and RMSE of estimators with omission both decrease as the sample size increases for all parameter configurations. Fifth, the JS procedure with omission dominates with respect to bias in all cases. It is also the dominant procedure in terms of RMSE unless the signal from the explosive regime is weak and the sample size is small in which case the PSY estimator yields a slightly smaller RMSE.

4.3 Results for the Two Bubbles Case

Tables 4-6 present results for the two bubbles case given by DGP-2. In addition to the five dating estimators considered in the single bubble case, we also include results for the HLW estimator. Table 4 reports the break date selection probabilities for each of the four break dates. Consistent with the single bubble case, the results show (i) the relatively low accuracy of all procedures in dating the origination of explosive behavior; (ii) that for each break, the JS/PQ procedures without omission tend to select a break date later than the corresponding true date; (iii) that the HLS estimates improve upon the PSY estimates, as expected; (iv) the JS procedure with omission can detect the implosion dates with highest accuracy for each of the sample sizes.

Table 5 reports the bias and RMSE of the break fraction estimators for each of the six procedures. The findings again indicate that the JS/PQ estimators without omission are subject to considerable biases which are not mitigated with a larger sample size. In contrast, their counterparts with omission are much more accurate in terms of both bias and RMSE, with the JS estimator incurring the smallest bias (RMSE) in most (all) cases. While the bias of the HLW estimator is comparable to that of the JS estimator with omission, the latter is more accurate in terms of RMSE for dating the onset and collapse of both bubbles.

Table 6 presents the bias and RMSE of the autoregressive coefficient estimators in the two explosive regimes. The estimates in both regimes are typically downward biased although the JS estimates with omission are virtually unbiased when $T = 400$. These results are strongly indicative of the consistency (inconsistency) of the JS/PQ estimators with (without) omission and clearly point to the superiority of the JS procedure. While the HLW estimator and the JS estimator with omission have similar accuracy (in terms of both bias and RMSE) with respect to the first bubble, the latter estimator dominates for the second bubble, particularly in terms of RMSE.

The results with serially correlated errors in both the single and two bubbles cases are overall qualitatively similar to those with i.i.d. errors with the dominance of the JS procedure that incorporates omission over the other procedures being even more evident under serial correlation, particularly with respect to estimation of the break dates and the autoregressive coefficients. Tables B.1-B.12 in Appendix B provide more detailed results.

4.4 Local versus Global Minimum

This subsection illustrates that the Perron and Qu (2006) algorithm often yields a larger value for the minimized sum of squared residuals than the proposed joint search algorithm, indicating that the estimated break dates obtained from the PQ algorithm may only correspond to a local instead of the global minimum. To this end, we present results on the difference between the sum of squared residuals based on the break date estimates from the JS and PQ procedures with omission, with a similar pattern holding for the corresponding procedures without omission. For example, in the single bubble case, after obtaining the date estimates $(\widetilde{T}_1^{JS}, \widetilde{T}_2^{JS})$ and $(\widetilde{T}_1^{PQ}, \widetilde{T}_2^{PQ})$, we compute

$$
\begin{aligned}
SSR_{om}^{JS} &= \sum_{t=2}^{\widetilde{T}_1^{JS}} (\Delta y_t)^2 + \sum_{t=\widetilde{T}_1^{JS}+1}^{\widetilde{T}_2^{JS}} (y_t - \widetilde{c}^{JS} - \widetilde{\delta}^{JS} y_{t-1})^2 + \sum_{t=\widetilde{T}_2^{JS}+2}^{T} (\Delta y_t)^2, \\
SSR_{om}^{PQ} &= \sum_{t=2}^{\widetilde{T}_1^{PQ}} (\Delta y_t)^2 + \sum_{t=\widetilde{T}_1^{PQ}+1}^{\widetilde{T}_2^{PQ}} (y_t - \widetilde{c}^{PQ} - \widetilde{\delta}^{PQ} y_{t-1})^2 + \sum_{t=\widetilde{T}_2^{PQ}+2}^{T} (\Delta y_t)^2,
\end{aligned}
$$

and their scaled difference

$$\Delta = T^{-1}\left(SSR_{om}^{JS} - SSR_{om}^{PQ}\right).$$

We analyze this quantity graphically as follows. We sort the differences from smallest to largest (across the replications) and include a vertical line showing the x-th replication at which Δ is exactly 0 (i.e., both estimators find the same minimum). To the left of this line, we have $SSR_{om}^{JS} < SSR_{om}^{PQ}$ while to the right, we have $SSR_{om}^{JS} = SSR_{om}^{PQ}$. The reference line $y = 0$ in each plot shows that Δ never exceeds 0, i.e., $SSR_{om}^{JS} \leq SSR_{om}^{PQ}$ always holds.

Figure 1 plots the results for DGP-1 with i.i.d. errors, where the break locations $(\lambda_1^0, \lambda_2^0) \in \{(0.5, 0.65), (0.4, 0.6)\}$. Figure 2 presents similar results for DGP-2. Specifically, we consider two break location configurations: $\lambda_{S1}^0 = (0.2, 0.3, 0.7, 0.8)$ and $\lambda_{S2}^0 = (0.2, 0.4, 0.6, 0.8)$. The patterns for the other seven break location configurations are similar and hence omitted.

The findings can be summarized as follows. First, the number of replications in which SSR_{om}^{JS} is strictly smaller than SSR_{om}^{PQ} can be considerable, as indicated by the location of the vertical line in each figure. Second, the maximal difference between SSR_{om}^{JS} and SSR_{om}^{PQ} can also be substantial, as indicated by the scale of the y-axis. Third, the number of replications for which PQ is unable to find the global minimum decreases as δ and/or T increases, as expected.

Finally, we also compared the average (over 5,000 replications) computing time incurred by the JS and PQ procedures to obtain the break date estimates and did not find any notable difference between them. Thus, JS not only delivers estimates with better statistical properties than PQ but is also computationally efficient.

4.5 Number of Bubbles

The application of the proposed date-stamping procedure as well as that of HLW relies on the PSY estimate of the number of bubbles in the initial step. The simulation results reported in the previous sections were obtained by conditioning on replications in which the PSY procedure selects the true number of bubbles. The Monte Carlo simulations reported in PSY show that their procedure delivers a reliable estimate of the number of bubbles for the types of bubble DGPs considered in our simulation design. In practice, however, the PSY approach may underestimate/overestimate the number of bubbles and it is thus of interest to examine the properties of dating estimators in these cases. Accordingly, using the same design as before, we now eliminate the replications in which the PSY algorithm selects the true number of bubbles and increase the number of replications until we obtain 1,000 replications in which the algorithm either underestimates the number of bubbles (i.e., selects one bubble under DGP-2) or overestimates it (i.e., selects two bubbles under DGP-1).

Tables B.13 and B.14 in Appendix B present the mean and standard deviation of the break fraction estimators, respectively, for the case where the number of bubbles is overestimated. Table B.15 reports the corresponding results in the underestimated case. In either case, the findings suggest that the dating estimators are inconsistent with their bias and standard deviation displaying no general tendency to decrease as the sample size increases. The accuracy of the date-stamping procedures thus crucially depends on whether the true number of bubbles is selected in the first step.

5 Empirical Illustration

The unprecedented surge in crude oil prices between 2003 and 2008 and its subsequent collapse during the Global Financial Crisis has been a subject of extensive debate and discussion among academics and policymakers. The West Texas Intermediate (WTI) price, regarded as one of the principal benchmarks for crude oil based on quality and location, rose from below \$30 per barrel at the beginning of 2003 to about \$147 in mid-July 2008 followed by a dramatic collapse to below \$40 in December 2008. A substantial body of research has been devoted to studying the major determinants of oil price fluctuations over this period. Kilian (2009) adopted a structural vector autoregressive modeling approach to show that oil price shocks are primarily driven by a combination of aggregate demand and pre-cautionary demand shocks with a minor contribution from supply shocks, while Kilian and Murphy (2014) found a limited role for speculative demand shocks in explaining crude oil price movements. In contrast, Hamilton (2009) attributed the sharp spike in oil prices between 2007-08 to a combination of demand shocks and stagnation in world production over 2005-07 but suggested that the ensuing collapse may be consistent with the bursting of a speculative bubble. Shi and Arora (2012) and Tsvetanov et al. (2016) provided evidence in favor of a rational bubble in crude oil prices, in accordance with the increased financialization of the oil futures markets and the expansion of index trading since 2004. More recently, Pavlidis et al. (2018) exploited the fact that in the presence of a speculative bubble, the difference between the future spot price and the expected price is explosive regardless of whether the fundamental component is explosive. They apply the PWY and PSY testing procedures to this difference to conclude against the presence of speculative bubbles over the period 1990-2013. In our empirical analysis, we do not take a stand on whether the explosive behavior in crude oil prices is driven by fundamental or speculative factors. Rather, we use the crude oil price series to illustrate the efficacy of our proposed approach in date-stamping the origination and termination of explosive behavior between 2003-2008.

Our analysis is based on the monthly real crude oil price computed as the nominal WTI price deflated by the U.S. consumer price index (CPI). The data are obtained from the FRED-MD database maintained by the Federal Reserve Bank of St. Louis and span the period from January 1986 to July 2014 so that the sample size is 343. The start date is chosen to avoid the period of regulation of the WTI market until the early 1980s while the end date is chosen to avoid the short period of a rapid price decline between late 2014 and early 2015 as well as the highly volatile regime between early 2020 and late 2022 during the COVID pandemic. Figure 3 plots the real oil price along with the sequence of $BSADF_{r_2}(r_0)$ statistics and the corresponding sequence of 95% critical values, where $r_2 \in [r_0, 1]$ and $r_0 = 0.01 + 1.8T^{-1/2}$ [see (14)-(15)]. The plot indicates that the oil price was characterized by relatively mild fluctuations until 2003 followed by a distinct run-up until 2008 with a sharp spike over the period 2007-2008 culminating in a dramatic collapse in July 2008. The PSY dating algorithm found a single statistically significant episode of explosive behavior between October 2007 and August 2008. Shorter periods of explosiveness were ruled out by the requirement that their duration be at least $\lfloor \ln T \rfloor = 5$ observations.

Next, we report the results of two ex post tests for explosiveness conducted for the full sample period. The first corresponds to the testing strategy recommended by PWY and entails taking the maximum of the ADF statistic sequence computed over a forward expanding sample of observations. Specifically, the test statistic is given by

$$SADF(r_0) = \sup_{r_2 \in [r_0, 1]} \{ADF_0^{r_2}\},$$

where $ADF_0^{r_2}$ is the ADF statistic based on observations in the range $[0, r_2]$. The second test statistic was proposed by PSY and is based on taking the maximum of the $BSADF_{r_2}(r_0)$ sequence over $r_2 \in [r_0, 1]$. Thus, the statistic is given by

$$GSADF(r_0) = \sup_{r_2 \in [r_0, 1]} \{BSADF_{r_2}(r_0)\}.$$

As shown in PSY, $GSADF(r_0)$ offers a more powerful testing strategy than $SADF(r_0)$ with multiple bubbles due to its double recursive nature that allows flexible window widths while $SADF(r_0)$ fixes the starting point of the recursion on the first observation. The results, presented in Panel A of Table 7, indicate that the null hypothesis of a unit root is rejected by the $SADF$ test at the 5% level and the $GSADF$ test at the 1% level, thereby confirming the presence of explosive behavior over the full sample.

We now turn to the results obtained from applying the different date-stamping methods. Panel B of Table 7 presents the estimated regimes obtained by fitting a single bubble model using each of the methods described in Section 4.1. The trimming level is set to 10% for the JS and PQ procedures. The JS procedure with omission dates the origination and termination of explosive behavior in September 2003 and August 2008, respectively. The estimated origination date was associated with a period of rapid economic growth in the OECD countries as well as China leading to an increase in the global demand for oil. Kilian (2009) used a historical decomposition of the real oil price based on a structural vector autoregression to show that the cumulative effect of aggregate demand shocks on the real oil price started to increase in late 2003. The estimated implosion date corresponds to the unfolding of the Global Financial Crisis that was associated with a sharp decrease in the global demand for oil. In comparison, the JS procedure without omission estimates the origination and termination dates as July 2008 and July 2011, respectively. These estimates are consistent with the theoretical analysis presented in Section 2. Specifically, the estimated start date of July 2008 is only a month prior to the collapse date estimated by the JS procedure with omission which is in close accordance with the prediction in Theorem 1 that the OLS estimate of the start date converges to the true implosion date in large samples. Similarly, the duration between the estimated end dates from the JS procedure with and without omission is 35 months, quite close to the large sample prediction of $\lfloor \epsilon T \rfloor = \lfloor 0.1 \times 343 \rfloor = 34$ months. A graphical comparison of the JS date estimates with and without omission is presented in Figure 4. The PQ procedure with omission estimates the explosive regime to run from July 1990 to August 2008. To our knowledge, such an extended duration of explosive behavior has little theoretical or empirical support in the literature. The PQ estimator without omission estimates the start and end dates of explosive behavior at September 2008 and June 2011, respectively, again in compliance with the prediction in Theorem 1. Finally, the PSY procedure identifies the explosive episode as lasting from October 2007 to August 2008 corresponding to the sharp spike in the oil price between 2007-2008 but fails to detect exuberance in the period prior to 2007.

Panel B of Table 7 also reports the results of the $SADF$ and $GSADF$ tests conducted within each of these regimes. The rejection pattern of these tests provide evidence on whether the estimated regimes are in fact consistent with the single bubble DGP in (1). Thus, if the first and third estimated regimes are in fact $I(1)$, the tests can be expected to fail to reject the unit root null in these regimes while if the second estimated regime is in fact explosive, the tests can be expected to reject. The rejection pattern for the JS and PQ procedures with omission are consistent with the single bubble specification although, as noted above, the estimated explosive regime for the latter is implausibly long. For each of the other three date-stamping procedures, the rejection pattern by at least one of the tests does not conform to a single bubble model. For instance, the JS procedure without omission finds evidence of explosiveness in the first estimated regime from January 1986 to June 2008 but not in the subsequent regimes, a pattern that is consistent with the prediction in Theorem 1.

Finally, we examine the sensitivity of the different date-stamping methods to variations in the sample period. Specifically, we consider subsamples that remove the first and/or the last six months from the full sample. The results, presented in Table 8, demonstrate the robustness of the JS procedure with omission. In particular, while the date estimates from the other approaches tend to vary with the sample period, the proposed approach delivers the same date estimates regardless of the sample period considered. While this evidence is consistent with the fact the proposed approach yields estimates that minimize the global sum of squared residuals, the erratic behavior of the PQ estimates with omission is consistent with the fact that these estimates may be susceptible to the problem of local minima.

6 Conclusion

This paper studies the properties of least squares estimates of the parameters in autoregressive models that involve switches between unit root and explosive regimes, where each explosive regime is followed by an implosion before the re-emergence of a unit root regime. It is shown that standard OLS estimators of the break dates/fractions and autoregressive coefficients are inconsistent due to their failure in properly accounting for the implosion points. A simple modification in the form of omitting the residuals corresponding to the potential implosion points when estimating the parameters restores consistency of the estimators. We also develop an efficient dynamic programming algorithm that facilitates estimation of the break dates without being susceptible to the problem of local minima, unlike the Perron and Qu (2006) iterative scheme based on initial values obtained from unrestricted Bai and Perron (2003) estimation. Monte Carlo simulations and an empirical application are used to provide support for our proposed method.

Data Availability Statement The data used in this article are obtained from the FRED-MD database maintained by the Federal Reserve Bank of St. Louis at https://www.stlouisfed.org/research/economists/mccracken/fred-databases.

References

Bai J, Perron P. 2003. Computation and analysis of multiple structural change models. *Journal of Applied Econometrics* **18**(1):1–22.

Brown RL, Durbin J, Evans JM. 1975. Techniques for Testing the Constancy of Regression Relationships over Time. *Journal of the Royal Statistical Society, Series B (Methodological)* **37**(2):149–192.

Hamilton JD. 2009. Causes and Consequences of the Oil Shock of 2007–08. *Brookings Papers on Economic Activity* **2009**:215–261.

Harvey DI, Leybourne SJ, Sollis R. 2017. Improving the accuracy of asset price bubble start and end date estimators. *Journal of Empirical Finance* **40**:121–138.

Harvey DI, Leybourne SJ, Whitehouse EJ. 2020. Date-stamping multiple bubble regimes. *Journal of Empirical Finance* **58**:226–246.

Hu Y. 2023. A review of Phillips-type right-tailed unit root bubble detection tests. *Journal of Economic Surveys* **37**(1):141–158.

Kejriwal M, Perron P, Zhou J. 2013. Wald tests for detecting multiple structural changes in persistence. *Econometric Theory* **29**(2):289–323.

Kilian L. 2009. Not all oil price shocks are alike: Disentangling demand and supply shocks in the crude oil market. *American Economic Review* **99**(3):1053–1069.

Kilian L, Murphy DP. 2014. The role of inventories and speculative trading in the global market for crude oil. *Journal of Applied Econometrics* **29**(3):454–478.

Kurozumi E. 2021. Asymptotic behavior of delay times of bubble monitoring tests. *Journal of Time Series Analysis* **42**(3):314–337.

Kurozumi E, Skrobotov A. 2023. On the asymptotic behavior of bubble date estimators. *Journal of Time Series Analysis* **44**(4):359–373.

Pang T, Du L, Chong TTL. 2021. Estimating multiple breaks in nonstationary autoregressive models. *Journal of Econometrics* **221**(1):277–311.

Pavlidis EG, Paya I, Peel DA. 2018. Using market expectations to test for speculative bubbles in the crude oil market. *Journal of Money, Credit and Banking* **50**(5):833–856.

Perron P, Qu Z. 2006. Estimating restricted structural change models. *Journal of Econometrics* **134**(2):373–399.

Phillips PC, Magdalinos T. 2007. Limit theory for moderate deviations from a unit root. *Journal of Econometrics* **136**(1):115–130.

Phillips PC, Shi S, Yu J. 2015a. Testing for multiple bubbles: Historical episodes of exuberance and collapse in the S&P 500. *International Economic Review* **56**(4):1043–1078.

Phillips PC, Shi S, Yu J. 2015b. Testing for multiple bubbles: Limit theory of real-time detectors. *International Economic Review* **56**(4):1079–1134.

Phillips PC, Shi SP. 2018. Financial bubble implosion and reverse regression. *Econometric Theory* **34**(4):705–753.

Phillips PC, Wu Y, Yu J. 2011. Explosive behavior in the 1990s Nasdaq: When did exuberance escalate asset values? *International Economic Review* **52**(1):201–226.

Phillips PC, Yu J. 2009. Limit theory for dating the origination and collapse of mildly explosive periods in time series data. *Unpublished manuscript, Sim Kee Boon Institute for Financial Economics, Singapore Management University* .

Shi S, Arora V. 2012. An application of models of speculative behaviour to oil prices. *Economics letters* **115**(3):469–472.

Skrobotov A. 2023. Testing for explosive bubbles: a review. *Dependence Modeling* **11**(1):20220152.

Tsvetanov D, Coakley J, Kellard N. 2016. Bubbling over! The behaviour of oil futures along the yield curve. *Journal of Empirical Finance* **38**:516–533.

Table 1: Probabilities of break date selection (single bubble)

Panel A: First date estimates												
δ	T	T_1^0	$\widehat{p}_1^C(JS)$	$\widehat{p}_1^L(JS)$	$\widehat{p}_1^C(PQ)$	$\widehat{p}_1^L(PQ)$	$\widetilde{p}_1^C(JS)$	$\widetilde{p}_1^L(JS)$	$\widetilde{p}_1^C(PQ)$	$\widetilde{p}_1^L(PQ)$	$\widehat{p}_1^C(PSY)$	$\widehat{p}_1^L(PSY)$
1.02	100	50	0.00	0.79	0.00	0.70	**0.03**	0.35	0.02	0.38	0.02	0.56
1.02	200	100	0.00	0.92	0.00	0.90	**0.03**	0.46	0.01	0.46	0.01	0.73
1.02	400	200	0.00	0.97	0.00	0.97	**0.04**	0.56	0.02	0.47	0.00	0.86
1.02	100	40	0.00	0.88	0.00	0.84	**0.03**	0.47	0.02	0.51	0.02	0.73
1.02	200	80	0.00	0.96	0.00	0.95	**0.03**	0.54	0.01	0.58	0.01	0.83
1.02	400	160	0.00	0.99	0.00	0.99	**0.03**	0.63	0.01	0.57	0.00	0.93
1.05	100	50	0.00	0.95	0.00	0.91	**0.08**	0.38	0.04	0.38	0.02	0.77
1.05	200	100	0.00	0.99	0.00	0.99	**0.11**	0.52	0.05	0.39	0.01	0.91
1.05	400	200	0.00	1.00	0.00	1.00	**0.15**	0.53	0.07	0.39	0.01	0.95
1.05	100	40	0.00	0.98	0.00	0.97	**0.07**	0.47	0.03	0.51	0.01	0.90
1.05	200	80	0.00	1.00	0.00	0.99	**0.07**	0.55	0.03	0.49	0.01	0.95
1.05	400	160	0.00	1.00	0.00	1.00	**0.12**	0.57	0.06	0.43	0.01	0.96
Panel B: Second date estimates												
δ	T	T_2^0	$\widehat{p}_1^C(JS)$	$\widehat{p}_2^L(JS)$	$\widehat{p}_2^C(PQ)$	$\widehat{p}_2^L(PQ)$	$\widetilde{p}_2^C(JS)$	$\widetilde{p}_2^L(JS)$	$\widetilde{p}_2^C(PQ)$	$\widetilde{p}_2^L(PQ)$	$\widehat{p}_2^C(PSY)$	$\widehat{p}_2^L(PSY)$
1.02	100	65	0.00	0.89	0.10	0.79	**0.64**	0.26	0.51	0.40	0.28	0.19
1.02	200	130	0.00	0.96	0.04	0.92	**0.85**	0.11	0.64	0.33	0.49	0.12
1.02	400	260	0.00	0.98	0.01	0.97	**0.96**	0.03	0.78	0.21	0.74	0.06
1.02	100	60	0.00	0.94	0.07	0.88	**0.70**	0.24	0.52	0.43	0.29	0.21
1.02	200	120	0.00	0.97	0.02	0.96	**0.89**	0.08	0.60	0.38	0.55	0.11
1.02	400	240	0.00	0.99	0.01	0.99	**0.97**	0.02	0.71	0.28	0.82	0.06
1.05	100	65	0.00	0.97	0.05	0.92	**0.88**	0.10	0.70	0.28	0.71	0.07
1.05	200	130	0.00	0.99	0.00	0.99	**0.98**	0.02	0.87	0.13	0.91	0.03
1.05	400	260	0.00	1.00	0.00	1.00	**1.00**	0.00	0.98	0.02	0.87	0.12
1.05	100	60	0.00	0.99	0.02	0.97	**0.93**	0.06	0.66	0.33	0.79	0.06
1.05	200	120	0.00	1.00	0.00	1.00	**0.99**	0.01	0.80	0.20	0.93	0.03
1.05	400	240	0.00	1.00	0.00	1.00	**1.00**	0.00	0.97	0.03	0.80	0.20

Note: 1) The superscript 'C' denotes the probability of correctly selecting the true break date.
2) The superscript 'L' denotes the probability of selecting a date later than the true break date.
3) The notation "ˆ" indicates "no omission" and "˜" indicates "with omission".
4) The method with the highest probability of correctly selecting the true break date is highlighted in bold.

Table 2: Bias and RMSE of break fraction estimates (single bubble)

δ	λ_1^0	λ_2^0	T	$\widehat{\lambda}_1^{JS}$	$\widehat{\lambda}_1^{PQ}$	$\widetilde{\lambda}_1^{JS}$	$\widetilde{\lambda}_1^{PQ}$	$\widehat{\lambda}_1^{PSY}$	$\widehat{\lambda}_2^{JS}$	$\widehat{\lambda}_2^{PQ}$	$\widetilde{\lambda}_2^{JS}$	$\widetilde{\lambda}_2^{PQ}$	$\widehat{\lambda}_2^{PSY}$
						Panel A: Bias							
1.02	0.5	0.65	100	0.077	0.036	−0.092	−0.087	**0.021**	0.080	0.078	**0.021**	0.048	−0.047
1.02	0.5	0.65	200	0.122	0.113	−0.054	−0.053	**0.036**	0.062	0.077	**0.007**	0.040	−0.045
1.02	0.5	0.65	400	0.140	0.139	**−0.006**	−0.030	0.051	0.052	0.063	**−0.000**	0.019	−0.027
1.02	0.4	0.6	100	0.160	0.137	−0.019	**0.004**	0.106	0.096	0.106	0.031	0.069	**−0.009**
1.02	0.4	0.6	200	0.185	0.182	**−0.007**	0.025	0.097	0.064	0.083	**0.007**	0.051	−0.023
1.02	0.4	0.6	400	0.196	0.195	**0.020**	0.023	0.092	0.055	0.065	**0.002**	0.025	−0.007
1.05	0.5	0.65	100	0.134	0.117	−0.052	−0.060	**0.038**	0.088	0.106	**0.010**	0.041	−0.022
1.05	0.5	0.65	200	0.147	0.146	**0.001**	−0.034	0.051	0.063	0.083	**0.001**	0.015	−0.008
1.05	0.5	0.65	400	0.149	0.149	**0.009**	−0.015	0.038	0.054	0.064	**0.000**	0.002	−0.002
1.05	0.4	0.6	100	0.193	0.188	−0.019	**0.009**	0.092	0.089	0.119	0.008	0.055	**−0.001**
1.05	0.4	0.6	200	0.199	0.198	0.014	**0.007**	0.079	0.064	0.087	0.001	0.024	**0.000**
1.05	0.4	0.6	400	0.200	0.200	0.017	**−0.003**	0.049	0.054	0.063	**0.000**	0.003	0.001
						Panel B: RMSE							
1.02	0.5	0.65	100	**0.183**	0.191	0.207	0.214	0.200	0.158	0.157	**0.125**	0.133	0.205
1.02	0.5	0.65	200	**0.164**	0.167	0.166	0.192	0.182	0.107	0.119	**0.075**	0.102	0.185
1.02	0.5	0.65	400	0.155	0.156	**0.097**	0.151	0.145	0.076	0.084	**0.044**	0.065	0.141
1.02	0.4	0.6	100	0.207	0.200	**0.169**	0.182	0.221	0.149	0.154	**0.112**	0.140	0.193
1.02	0.4	0.6	200	0.202	0.201	**0.138**	0.172	0.187	0.096	0.111	**0.063**	0.108	0.160
1.02	0.4	0.6	400	0.201	0.201	**0.091**	0.150	0.148	0.069	0.079	**0.036**	0.063	0.107
1.05	0.5	0.65	100	0.160	0.161	0.149	0.185	**0.137**	0.121	0.141	**0.068**	0.102	0.131
1.05	0.5	0.65	200	0.152	0.152	**0.068**	0.128	0.100	0.073	0.099	**0.029**	0.055	0.081
1.05	0.5	0.65	400	0.151	0.151	**0.039**	0.066	0.067	0.057	0.070	**0.014**	0.019	0.048
1.05	0.4	0.6	100	0.202	0.200	**0.125**	0.165	0.142	0.113	0.146	**0.054**	0.112	0.099
1.05	0.4	0.6	200	0.200	0.200	**0.069**	0.130	0.108	0.072	0.101	**0.021**	0.063	0.057
1.05	0.4	0.6	400	0.200	0.200	**0.044**	0.066	0.072	0.057	0.068	**0.011**	0.020	0.032

Note: 1) The notation “ˆ” indicates “no omission” and “˜” indicates “with omission”.
2) The best method is highlighted in bold.

Table 3: Bias and RMSE of AR(1) estimates (single bubble)

δ	λ_1^0	λ_2^0	T	$\widehat{\delta}^{JS}$	$\widehat{\delta}^{PQ}$	$\widetilde{\delta}^{JS}$	$\widetilde{\delta}^{PQ}$	$\widehat{\delta}^{PSY}$
					Panel A: Bias			
1.02	0.5	0.65	100	−0.858	−0.725	−0.315	−0.328	**−0.298**
1.02	0.5	0.65	200	−0.895	−0.842	**−0.169**	−0.274	−0.232
1.02	0.5	0.65	400	−0.918	−0.901	**−0.057**	−0.176	−0.122
1.02	0.4	0.6	100	−0.894	−0.801	−0.304	−0.369	**−0.287**
1.02	0.4	0.6	200	−0.935	−0.900	**−0.147**	−0.338	−0.205
1.02	0.4	0.6	400	−0.954	−0.943	**−0.041**	−0.254	−0.087
1.05	0.5	0.65	100	−1.006	−0.932	**−0.178**	−0.288	−0.178
1.05	0.5	0.65	200	−1.027	−1.016	**−0.051**	−0.131	−0.063
1.05	0.5	0.65	400	−1.036	−1.034	**−0.008**	−0.016	−0.034
1.05	0.4	0.6	100	−1.035	−0.993	**−0.144**	−0.348	−0.150
1.05	0.4	0.6	200	−1.037	−1.030	**−0.036**	−0.206	−0.043
1.05	0.4	0.6	400	−1.044	−1.044	**−0.004**	−0.031	−0.035
					Panel B: RMSE			
1.02	0.5	0.65	100	0.955	0.826	0.464	0.491	**0.381**
1.02	0.5	0.65	200	0.933	0.893	**0.292**	0.460	0.324
1.02	0.5	0.65	400	0.937	0.926	**0.132**	0.382	0.232
1.02	0.4	0.6	100	0.974	0.879	0.450	0.533	**0.372**
1.02	0.4	0.6	200	0.959	0.933	**0.268**	0.532	0.295
1.02	0.4	0.6	400	0.967	0.960	**0.108**	0.483	0.196
1.05	0.5	0.65	100	1.038	0.976	0.319	0.491	**0.286**
1.05	0.5	0.65	200	1.033	1.024	**0.135**	0.339	0.157
1.05	0.5	0.65	400	1.039	1.038	**0.041**	0.109	0.102
1.05	0.4	0.6	100	1.063	1.015	0.282	0.561	**0.256**
1.05	0.4	0.6	200	1.041	1.035	**0.108**	0.447	0.118
1.05	0.4	0.6	400	1.045	1.045	**0.033**	0.173	0.086

Note: 1) The notation "ˆ" indicates "no omission" and "˜" indicates "with omission".
2) The best method is highlighted in bold.

Table 4: Probabilities of break date selection (two bubbles)

δ_1	T	T_i^0	$\widehat{p}_i^C(JS)$	$\widehat{p}_i^L(JS)$	$\widehat{p}_i^C(PQ)$	$\widehat{p}_i^L(PQ)$	$\widetilde{p}_i^C(JS)$	$\widetilde{p}_i^L(JS)$	$\widetilde{p}_i^C(PQ)$	$\widetilde{p}_i^L(PQ)$	$\widehat{p}_i^C(PSY)$	$\widehat{p}_i^L(PSY)$	$\widehat{p}_i^C(HLW)$	$\widehat{p}_i^L(HLW)$
Panel A: First break date [i=1]														
1.05	100	20	0.00	0.98	0.01	0.92	**0.05**	0.45	0.03	0.60	0.02	0.93	**0.05**	0.44
1.05	200	40	0.00	1.00	0.00	0.99	**0.05**	0.58	0.02	0.64	0.01	0.96	**0.05**	0.57
1.05	400	80	0.00	0.99	0.00	1.00	**0.08**	0.63	0.03	0.58	0.01	0.97	**0.08**	0.63
Panel B: Second break date [i=2]														
1.05	100	40	0.00	0.98	0.10	0.88	**0.92**	0.06	0.53	0.41	0.66	0.08	0.87	0.06
1.05	200	80	0.00	0.99	0.01	0.98	**0.98**	0.02	0.57	0.39	0.91	0.04	0.97	0.01
1.05	400	160	0.00	0.99	0.00	1.00	**1.00**	0.00	0.73	0.25	0.80	0.19	0.99	0.00
δ_2	T	T_i^0	$\widehat{p}_i^C(JS)$	$\widehat{p}_i^L(JS)$	$\widehat{p}_i^C(PQ)$	$\widehat{p}_i^L(PQ)$	$\widetilde{p}_i^C(JS)$	$\widetilde{p}_i^L(JS)$	$\widetilde{p}_i^C(PQ)$	$\widetilde{p}_i^L(PQ)$	$\widehat{p}_i^C(PSY)$	$\widehat{p}_i^L(PSY)$	$\widehat{p}_i^C(HLW)$	$\widehat{p}_i^L(HLW)$
Panel C: Third break date [i=3]														
1.05	100	60	0.00	0.98	0.01	0.87	**0.09**	0.51	0.04	0.61	0.01	0.91	0.08	0.48
1.05	200	120	0.00	0.99	0.00	0.98	**0.10**	0.56	0.04	0.59	0.01	0.95	**0.10**	0.54
1.05	400	240	0.00	0.99	0.00	1.00	**0.12**	0.57	0.05	0.51	0.00	0.98	**0.12**	0.56
Panel D: Fourth break date [i=4]														
1.05	100	80	0.00	0.98	0.19	0.80	**0.95**	0.04	0.65	0.34	0.82	0.01	0.92	0.04
1.05	200	160	0.00	0.99	0.02	0.97	**0.99**	0.01	0.70	0.29	0.94	0.02	0.98	0.01
1.05	400	320	0.00	0.99	0.00	1.00	**1.00**	0.00	0.83	0.16	0.81	0.19	**1.00**	0.00

Note: See notes to Table 1.

Table 5: Bias and RMSE of break fraction estimates (two bubbles)

δ_1	λ_1^0	λ_2^0	T	$\widehat{\lambda}_1^{JS}$	$\widehat{\lambda}_2^{JS}$	$\widetilde{\lambda}_1^{PQ}$	$\widetilde{\lambda}_2^{PQ}$	$\widetilde{\lambda}_1^{JS}$	$\widetilde{\lambda}_2^{JS}$	$\widetilde{\lambda}_1^{PQ}$	$\widetilde{\lambda}_2^{PQ}$	$\widehat{\lambda}_1^{PSY}$	$\widehat{\lambda}_2^{PSY}$	$\widehat{\lambda}_1^{HLW}$	$\widehat{\lambda}_2^{HLW}$
Panel A: Bias of first bubble estimators															
1.05	0.2	0.4	100	0.200	0.090	0.170	0.100	0.000^s	0.010	0.060	0.060	0.110	0.000^c	0.008	0.000^c
1.05	0.2	0.4	200	0.200	0.070	0.200	0.090	0.020^s	0.000^c	0.070	0.040	0.090	0.000^c	0.021	0.000^c
1.05	0.2	0.4	400	0.200	0.060	0.200	0.060	0.020^s	0.000^c	0.050	0.020	0.060	0.000^c	0.020^s	0.000^c
Panel B: RMSE of first bubble estimators															
1.05	0.2	0.4	100	0.200	0.120	0.190	0.130	0.090^s	0.050^c	0.140	0.120	0.140	0.090	0.093	0.053
1.05	0.2	0.4	200	0.200	0.080	0.200	0.100	0.070^s	0.020^c	0.140	0.100	0.120	0.050	0.081	0.034
1.05	0.2	0.4	400	0.210	0.080	0.200	0.070	0.050^s	0.010^c	0.110	0.080	0.070	0.020	0.050^s	0.010^c
δ_2	λ_3^0	λ_4^0	T	$\widehat{\lambda}_3^{JS}$	$\widehat{\lambda}_4^{JS}$	$\widetilde{\lambda}_3^{PQ}$	$\widetilde{\lambda}_4^{PQ}$	$\widetilde{\lambda}_3^{JS}$	$\widetilde{\lambda}_4^{JS}$	$\widetilde{\lambda}_3^{PQ}$	$\widetilde{\lambda}_4^{PQ}$	$\widehat{\lambda}_3^{PSY}$	$\widehat{\lambda}_4^{PSY}$	$\widehat{\lambda}_3^{HLW}$	$\widehat{\lambda}_4^{HLW}$
Panel C: Bias of second bubble estimators															
1.05	0.6	0.8	100	0.190	0.070	0.150	0.070	0.010^s	0.000^c	0.050	0.030	0.070	−0.030	0.010^s	−0.008
1.05	0.6	0.8	200	0.200	0.060	0.190	0.070	0.020^s	0.000^c	0.050	0.030	0.070	−0.010	0.020^s	−0.003
1.05	0.6	0.8	400	0.200	0.050	0.200	0.060	0.020^s	0.000^c	0.030	0.010	0.050	0.000^c	0.020^s	0.000^c
Panel D: RMSE of second bubble estimators															
1.05	0.6	0.8	100	0.200	0.090	0.180	0.090	0.080^s	0.030^c	0.140	0.060	0.120	0.110	0.093	0.052
1.05	0.6	0.8	200	0.200	0.070	0.200	0.080	0.070^s	0.020^c	0.130	0.050	0.100	0.070	0.072	0.034
1.05	0.6	0.8	400	0.200	0.070	0.200	0.070	0.040^s	0.000^c	0.100	0.040	0.070	0.020	0.044	0.018

Note: 1) The superscript 's' denotes bubble origination estimates with lowest bias/RMSE.
2) The superscript 'c' denotes bubble crash estimates with lowest bias/RMSE.
3) The notation "ˆ" indicates "no omission" and "˜" indicates "with omission".

Table 6: Bias and RMSE of AR(1) estimates (two bubbles)

δ	T	$\widehat{\delta}_1^{JS}$	$\widehat{\delta}_1^{PQ}$	$\widetilde{\delta}_1^{JS}$	$\widetilde{\delta}_1^{PQ}$	$\widehat{\delta}_1^{PSY}$	$\widehat{\delta}_1^{HLW}$	$\widehat{\delta}_2^{JS}$	$\widehat{\delta}_2^{PQ}$	$\widetilde{\delta}_2^{JS}$	$\widetilde{\delta}_2^{PQ}$	$\widehat{\delta}_2^{PSY}$	$\widehat{\delta}_2^{HLW}$
Panel A: Bias													
1.05	100	−1.010	−0.920	**−0.150**	−0.450	−0.450	**−0.150**	−1.050	−0.850	**−0.110**	−0.390	**−0.110**	−0.130
1.05	200	−1.020	−1.010	**−0.050**	−0.390	−0.390	**−0.050**	−1.040	−1.010	−0.040	−0.300	**−0.030**	**−0.030**
1.05	400	−1.010	−1.040	**−0.010**	−0.240	−0.240	**−0.010**	−1.050	−1.040	**0.000**	−0.150	−0.030	−0.010
Panel B: RMSE													
1.05	100	1.050	0.980	**0.290**	0.650	0.650	**0.290**	1.060	0.950	0.240	0.610	**0.200**	0.290
1.05	200	1.030	1.030	**0.130**	0.620	0.620	0.140	1.040	1.030	0.110	0.540	**0.100**	0.130
1.05	400	1.030	1.040	**0.030**	0.500	0.500	**0.030**	1.050	1.050	**0.020**	0.390	0.080	0.060

Note: See notes to Table 3.

Table 7: Tests for the Presence of Explosiveness

Panel A: Full Sample Tests						
		Sample	$SADF$	$\mathcal{P}_{SADF}$	$GSADF$	$\mathcal{P}_{GSADF}$
		1986:01−2014:07	2.024	0.012	3.221	0.006
Panel B: Estimated Regimes and Subsample Tests						
Procedure	Regime	Estimated regime	$SADF$	$\mathcal{P}_{SADF}$	$GSADF$	$\mathcal{P}_{GSADF}$
JS with	I(1)	1986:01−2003:08	−1.733	0.993	0.801	0.751
omission	I(e)	2003:09−2008:08	2.104	0.011	2.609	0.039
	I(1)	2008:09−2014:07	−1.063	0.884	1.098	0.433
JS without	I(1)	1986:01−2008:06	2.024	0.013	3.221	0.005
omission	I(e)	2008:07−2011:07	−2.180	0.982	1.173	0.316
	I(1)	2011:08−2014:07	−1.869	0.966	−0.795	0.987
PQ with	I(1)	1986:01−1990:06	−1.059	0.863	0.397	0.753
omission	I(e)	1990:07−2008:08	2.332	0.005	3.221	0.005
	I(1)	2008:09−2014:07	−1.063	0.884	1.098	0.433
PQ without	I(1)	1986:01−2008:08	2.024	0.012	3.220	0.004
omission	I(e)	2008:09−2011:06	2.332	0.005	3.221	0.005
	I(1)	2011:07−2014:07	−2.199	0.983	−0.795	0.987
PSY	I(1)	1986:01−2007:07	−0.662	0.864	1.017	0.649
	I(e)	2007:10−2008:08	3.729	0.026	3.729	0.136
	I(1)	2008:09−2014:07	−1.063	0.884	1.098	0.433

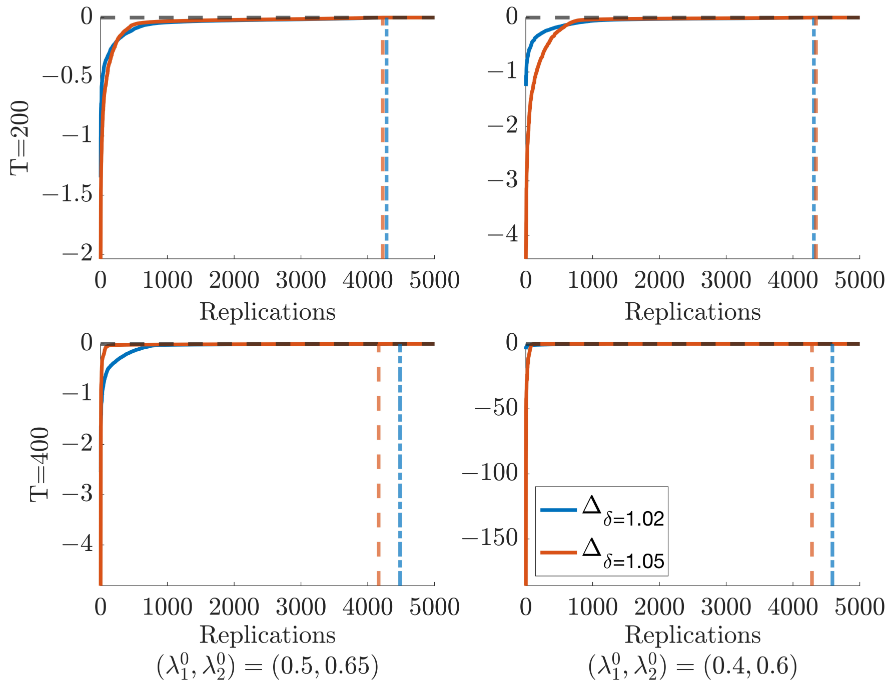
Note: 1) $SADF$, $GSADF$ denote the values of test statistics and $\mathcal{P}_{SADF}$, $\mathcal{P}_{GSADF}$ denote the corresponding p-values.
2) I(1) denotes a unit root regime and I(e) denotes an explosive regime.

Table 8: Estimated Explosive Regime of Real Oil Price (different subsamples)

Start	End	$\widetilde{T}_1^{JS}$	$\widetilde{T}_2^{JS}$	$\widehat{T}_1^{JS}$	$\widehat{T}_2^{JS}$	$\widetilde{T}_1^{PQ}$	$\widetilde{T}_2^{PQ}$	$\widehat{T}_1^{PQ}$	$\widehat{T}_2^{PQ}$	$\widehat{T}_1^{PSY}$	$\widehat{T}_2^{PSY}$
1986:01	2014:07	2003:09	2008:08	2008:07	2011:07	1990:07	2008:08	2008:09	2011:06	2007:10	2008:08
1986:01	2014:01	2003:09	2008:08	2008:05	2011:02	1990:07	2008:08	1990:07	2008:08	2007:10	2008:08
1986:07	2014:07	2003:09	2008:08	2008:07	2011:07	1990:07	2008:08	2008:09	2011:06	2007:10	2008:08
1986:07	2014:01	2003:09	2008:08	2008:07	2011:03	2008:05	2011:01	2008:05	2011:01	2007:10	2008:08
1987:01	2014:07	2003:09	2008:08	2008:07	2011:07	2008:09	2011:09	2008:09	2011:09	2007:10	2008:08
1987:01	2014:01	2003:09	2008:08	2008:07	2011:03	2008:05	2011:01	2008:05	2011:01	2007:10	2008:08

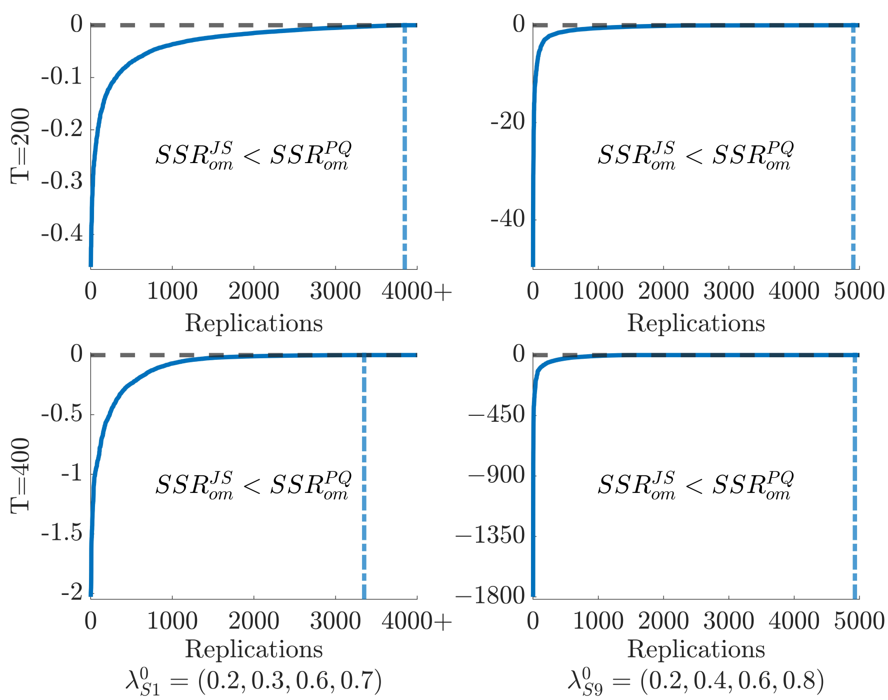
Note: The notation "ˆ" indicates "no omission" and "˜" indicates "with omission".

Figure 1: Difference between sums of squared residuals, $\Delta = T^{-1}(SSR^{JS}_{om} - SSR^{PQ}_{om})$ [Single Bubble Case]



Note: The differences are sorted from smallest to largest (across the replications). The vertical line indicates the replication at which $\Delta = 0$.

Figure 2: Difference between sums of squared residuals, $\Delta = T^{-1}(SSR^{JS}_{om} - SSR^{PQ}_{om})$ [Two Bubbles Case]



Note: The differences are sorted from smallest to largest (across the replications). The vertical line indicates the replication at which $\Delta = 0$.

Figure 3: Explosiveness in the Real Oil Price

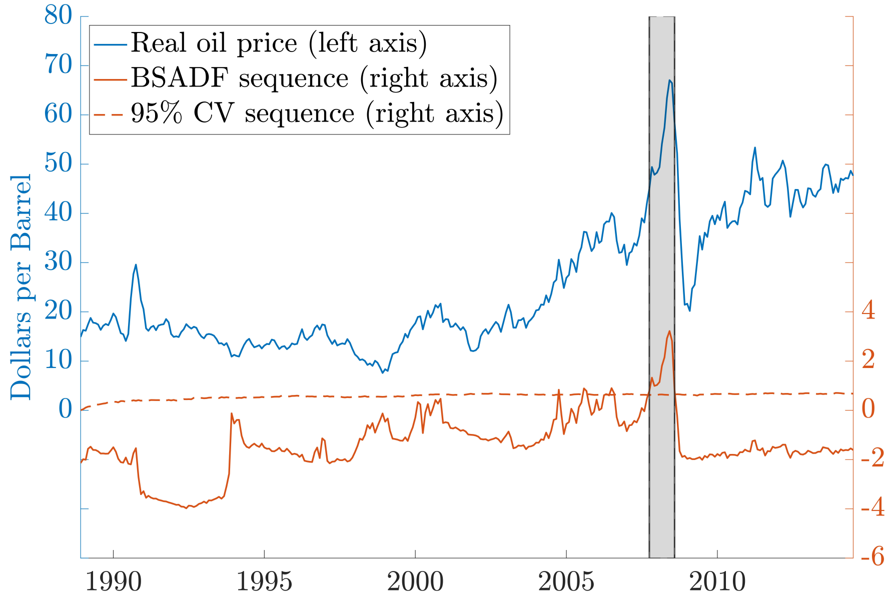
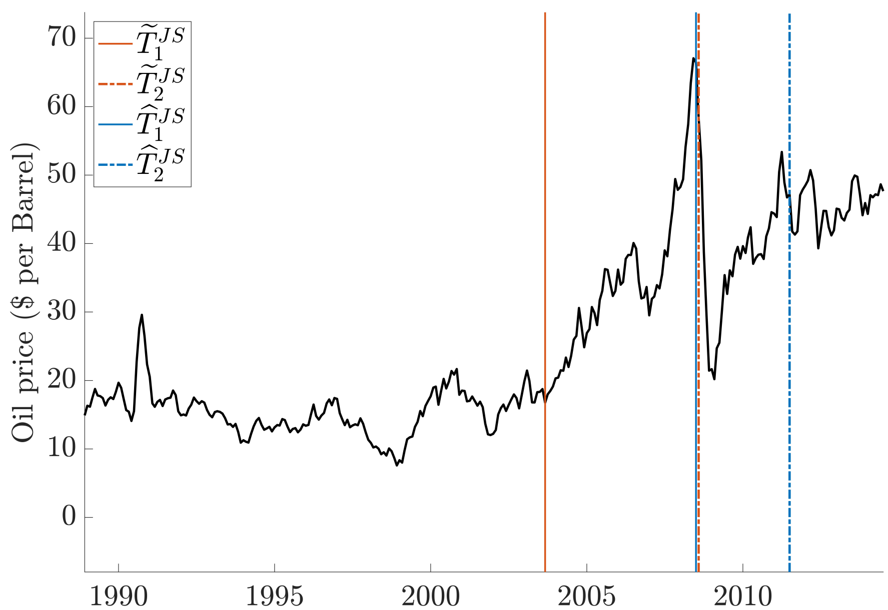


Figure 4: Estimated Dates with and without Omission



An Improved Procedure for Retrospectively Dating the Emergence and Collapse of Bubbles: Online Supplement

Mohitosh Kejriwal* *Purdue University*

Linh Nguyen† *Purdue University*

Pierre Perron‡ *Boston University*

Appendix A: Proofs of Theoretical Results

As a matter of notation, '$\xrightarrow{p}$' and '$\Rightarrow$' denote, respectively, convergence in probability and weak convergence of the associated probability measures. $W(.)$ denotes a standard Brownian motion on $[0,1]$ and $O_p^+(.)$ denotes a random quantity of the specified order that is asymptotically positive. For a random quantity z, we write $z = z_0 + o_p(z_0)$ as $z = z_0 + s.o.$, where $s.o.$ represents a term of smaller order in probability. The true break dates are denoted by $T_1^0 = \lfloor \lambda_1^0 T \rfloor$, $T_2^0 = \lfloor \lambda_2^0 T \rfloor$, where λ_1^0, λ_2^0 are the true break fractions. Following Harvey et al. (2017), we simplify the proofs by assuming that a constant is not included in the regression.

The data generating process (DGP) considered in the theoretical analysis is restated here for convenience:

$$y_t = \begin{cases} y_{t-1} + u_t, & 1 \le t \le T_1^0, \\ \delta y_{t-1} + u_t, & T_1^0 + 1 \le t \le T_2^0, \\ y_{T_1^0} + z^* + \sum_{j=T_2^0+1}^{t} u_j, & T_2^0 + 1 \le t \le T, \end{cases} \tag{A.1}$$

where u_t is i.i.d. with $E(u_t) = 0$, $E(u_t^2) = \sigma^2$ and $y_0 = o_p(T^{1/2})$, $z^* = O_p(1)$.

We first state two lemmas that will be used subsequently. The first follows from Lemma 1 in Harvey et al. (2017). The proof of the second follows from standard results for $I(1)$ processes [see, e.g., Perron (1989), Perron (1990)] and is thus omitted.

Lemma A.1 (Harvey et al. (2017)) *Assume that y_t is generated by (A.1). Let $S_T = T_1^0 \left[\delta^{2(T_2^0 - T_1^0)}\right]$. Then*

(a) $S_T/T \to \infty$;

(b) $y_{T_2^0}^2 = O_p^+(S_T)$;

(c) $S_T^{-1} \sum_{t=T_1^0+1}^{T_2^0} y_{t-1}^2 = (\delta^2 - 1)^{-1}\omega + o_p(1)$, *where* $\omega = \lim_{T\to\infty} S_T^{-1/2} y_{T_2^0}$;

*Daniels School of Business, Purdue University, 403 West State Street, West Lafayette IN 47906 (mkejriwa@purdue.edu).

†Daniels School of Business, Purdue University, 403 West State Street, West Lafayette IN 47906 (nguye535@purdue.edu).

‡Department of Economics, Boston University, 270 Bay State Road, Boston MA 02215 (perron@bu.edu).

(d) $\sum_{t=T_1^0+1}^{T_2^0} y_{t-1}u_t = O_p(S_T^{1/2})$.

Lemma A.2 *Assume that y_t is generated by (A.1). Then the following results hold jointly:*

(a) $T^{-1}\sum_{t=1}^{T_1^0} y_{t-1}u_t \Rightarrow \frac{\sigma^2(W^2(\lambda_1^0)-\lambda_1^0)}{2}$;

(b) $T^{-2}\sum_{t=1}^{T_1^0} y_{t-1}^2 \Rightarrow \sigma^2\int_0^{\lambda_1^0} W^2(r)dr$;

(c) $T^{-1/2}y_{T_1^0} \Rightarrow \sigma W(\lambda_1^0)$;

(d) $T^{-1/2}y_{T_2^0+1} \Rightarrow \sigma W(\lambda_1^0)$;

(e) $T^{-2}\sum_{t=T_2^0+2}^{T_2^0+\lfloor \varepsilon T\rfloor} y_{t-1}^2 \Rightarrow \sigma^2\int_0^{\varepsilon}[W(\lambda_1^0+r)]^2dr$;

(f) $T^{-1}\sum_{t=T_2^0+2}^{T_2^0+\lfloor \varepsilon T\rfloor} y_{t-1}u_t \Rightarrow \sigma^2\left[W(\lambda_1^0)\{W(\lambda_2^0+\epsilon)\} + \frac{\{W(\lambda_2^0+\epsilon)-W(\lambda_2^0)\}^2-\epsilon}{2}\right]$.

Proof of Theorem 1: (a) Given (T_1, T_2), the sum of squared residuals is given by

$$SSR(T_1,T_2) = \sum_{t=2}^{T_1}(\Delta y_t)^2 + \sum_{t=T_1+1}^{T_2}\{y_t - \hat{\delta}(T_1,T_2)y_{t-1}\}^2 + \sum_{t=T_2+1}^{T}(\Delta y_t)^2, \tag{A.2}$$

where

$$\hat{\delta}(T_1,T_2) = \left(\sum_{t=T_1+1}^{T_2} y_{t-1}^2\right)^{-1}\sum_{t=T_1+1}^{T_2} y_t y_{t-1}. \tag{A.3}$$

Defining $T_{2,\epsilon}^0 = T_2^0 + \lfloor \epsilon T\rfloor$, we have

$$\begin{aligned}\hat{\delta}(T_2^0,T_{2,\epsilon}^0) &= \left(\sum_{t=T_2^0+1}^{T_{2,\epsilon}^0} y_{t-1}^2\right)^{-1}\sum_{t=T_2^0+1}^{T_{2,\epsilon}^0} y_t y_{t-1}\\ &= \left(S_T^{-1}y_{T_2^0}^2 + S_T^{-1}\sum_{t=T_2^0+2}^{T_{2,\epsilon}^0} y_{t-1}^2\right)^{-1}\\ &\times\left(S_T^{-1}y_{T_2^0+1}y_{T_2^0} + S_T^{-1}\sum_{t=T_2^0+2}^{T_{2,\epsilon}^0} y_{t-1}^2 + S_T^{-1}\sum_{t=T_2^0+2}^{T_{2,\epsilon}^0} y_{t-1}u_t\right)\end{aligned}$$

and

$$S_T^{1/2}T^{-1/2}\hat{\delta}(T_2^0,T_{2,\epsilon}^0) = \left(S_T^{-1}y_{T_2^0}^2 + o_p(1)\right)^{-1}\left(T^{-1/2}y_{T_2^0+1}S_T^{-1/2}y_{T_2^0} + o_p(1)\right) \Rightarrow \omega^{-1}W(\lambda_1^0) = O_p(1), \tag{A.4}$$

where the last line follows using Lemmas A.1-A.2. Next, suppose that

$$\begin{aligned}\hat{T}_1 &= T_2^0 + k_1,\\ \hat{T}_2 &= T_{2,\epsilon}^0 + k_2,\end{aligned}$$

where k_1, k_2 are $O(1)$ integers. Let

$$\begin{aligned} F(k_1,k_2) &= SSR(\hat{T}_1,\hat{T}_2) - SSR(T_2^0, T_{2,\epsilon}^0) \\ &= \left[\sum_{t=2}^{\hat{T}_1}(\Delta y_t)^2 + \sum_{t=\hat{T}_1+1}^{\hat{T}_2}\{y_t - \hat{\delta}(\hat{T}_1,\hat{T}_2)y_{t-1}\}^2 + \sum_{t=\hat{T}_2+1}^{T}(\Delta y_t)^2\right] \\ &\quad - \left[\sum_{t=2}^{T_2^0}(\Delta y_t)^2 + \sum_{t=T_2^0+1}^{T_{2,\epsilon}^0}\{y_t - \hat{\delta}(T_2^0,T_{2,\epsilon}^0)y_{t-1}\}^2 + \sum_{t=T_{2,\epsilon}^0+1}^{T}(\Delta y_t)^2\right]. \end{aligned} \tag{A.5}$$

Now, the quantity $SSR(T_2^0, T_{2,\epsilon}^0)$ can be written as

$$\begin{aligned} SSR(T_2^0, T_{2,\epsilon}^0) &= \sum_{t=2}^{T} u_t^2 1(t \neq T_2^0+1) + (\delta-1)^2 \sum_{t=T_1^0+1}^{T_2^0} y_{t-1}^2 + 2(\delta-1)\sum_{t=T_1^0+1}^{T_2^0} y_{t-1}u_t \\ &\quad + \{y_{T_2^0+1} - \hat{\delta}(T_2^0,T_{2,\epsilon}^0)y_{T_2^0}\}^2 + (1-\hat{\delta}(T_2^0,T_{2,\epsilon}^0))^2 \sum_{t=T_2^0+2}^{T_{2,\epsilon}^0} y_{t-1}^2 + 2(1-\hat{\delta}(T_2^0,T_{2,\epsilon}^0))\sum_{t=T_2^0+2}^{T_{2,\epsilon}^0} y_{t-1}u_t \\ &= \sum_{t=2}^{T} u_t^2 1(t \neq T_2^0+1) + (\delta-1)^2 \sum_{t=T_1^0+1}^{T_2^0} y_{t-1}^2 + 2(\delta-1)\sum_{t=T_1^0+1}^{T_2^0} y_{t-1}u_t \\ &\quad + \sum_{t=T_2^0+2}^{T_{2,\epsilon}^0} y_{t-1}^2 + 2\sum_{t=T_2^0+2}^{T_{2,\epsilon}^0} y_{t-1}u_t + s.o., \end{aligned} \tag{A.6}$$

where the second equality follows from (A.4).

We will show that if either $k_1 \neq 0$ or $k_2 \neq 0$, then $F(k_1,k_2) > 0$ asymptotically. Note that when $k_2 = 0$, we must have $k_1 < 0$ due to the restriction $\hat{T}_2 - \hat{T}_1 \geq \lfloor \epsilon T \rfloor$. Similarly, if $k_1 = 0$, we must have $k_2 > 0$. We consider each of these cases in turn.

Case 1: $k_1 < 0,\ k_2 = 0$. First, observe that

$$\begin{aligned} \hat{\delta}(T_2^0+k_1, T_{2,\epsilon}^0) &= \left(\sum_{t=T_2^0+k_1+1}^{T_{2,\epsilon}^0} y_{t-1}^2\right)^{-1} \sum_{t=T_2^0+k_1+1}^{T_{2,\epsilon}^0} y_t y_{t-1} \\ &= \delta + \left(\sum_{t=T_2^0+k_1+1}^{T_{2,\epsilon}^0} y_{t-1}^2\right)^{-1}\left[-\delta y_{T_2^0}^2 + y_{T_2^0+1}y_{T_2^0} + (1-\delta)\sum_{t=T_2^0+2}^{T_{2,\epsilon}^0} y_{t-1}^2 + \sum_{t=T_2^0+2}^{T_{2,\epsilon}^0} y_{t-1}u_t\right] \\ &= \delta + \left(S_T^{-1}\sum_{t=T_2^0+k_1+1}^{T_2^0} y_{t-1}^2 + S_T^{-1}y_{T_2^0}^2 + o_p(1)\right)^{-1}\left[-\delta S_T^{-1} y_{T_2^0}^2 + o_p(1)\right] \quad \text{[by Lemmas A.1-A.2]} \\ &= \delta + \left(S_T^{-1}y_{T_2^0}^2\left\{(\delta^2-1)^{-1}(1-\delta^{-2|k_1|})+1\right\} + o_p(1)\right)^{-1}\left[-\delta S_T^{-1}y_{T_2^0}^2 + o_p(1)\right] \\ &= \delta - [\delta(1-\delta^{-2(|k_1|+1)})]^{-1}(\delta^2-1) + o_p(1) = \bar{\delta} + o_p(1) \xrightarrow{p} \bar{\delta} < 1/\delta, \end{aligned} \tag{A.7}$$

since $\bar{\delta} < \delta - \delta^{-1}(\delta^2 - 1) = 1/\delta$. Next, we can write

$$
\begin{aligned}
SSR(T_2^0 + k_1, T_{2,\epsilon}^0) &= \sum_{t=2}^{T_2^0+k_1} (\Delta y_t)^2 + \sum_{t=T_2^0+k_1+1}^{T_{2,\epsilon}^0} \{y_t - \hat{\delta}(T_2^0 + k_1, T_{2,\epsilon}^0) y_{t-1}\}^2 + \sum_{t=T_{2,\epsilon}^0+1}^{T} (\Delta y_t)^2 \\
&= \sum_{t=2}^{T_1^0} u_t^2 + \sum_{t=T_1^0+1}^{T_2^0+k_1} \{(\delta - 1) y_{t-1} + u_t\}^2 + \sum_{t=T_2^0+k_1+1}^{T_2^0} \{(\delta - \hat{\delta}(T_2^0 + k_1, T_{2,\epsilon}^0)) y_{t-1} + u_t\}^2 \\
&+ \{y_{T_2^0+1} - \hat{\delta}(T_2^0 + k_1, T_{2,\epsilon}^0) y_{T_2^0}\}^2 + \sum_{t=T_2^0+2}^{T_{2,\epsilon}^0} \{(1 - \hat{\delta}(T_2^0 + k_1, T_{2,\epsilon}^0)) y_{t-1} + u_t\}^2 \\
&+ \sum_{t=T_{2,\epsilon}^0+1}^{T} u_t^2 \\
&= \sum_{t=2}^{T} u_t^2 1(t \neq T_2^0 + 1) + (\delta - 1)^2 \sum_{t=T_1^0+1}^{T_2^0+k_1} y_{t-1}^2 + 2(\delta - 1) \sum_{t=T_1^0+1}^{T_2^0+k_1} y_{t-1} u_t \\
&+ (\delta - \bar{\delta})^2 \sum_{t=T_2^0+k_1+1}^{T_2^0} y_{t-1}^2 + 2(\delta - \bar{\delta}) \sum_{t=T_2^0+k_1+1}^{T_2^0} y_{t-1} u_t + \{y_{T_2^0+1} - \bar{\delta} y_{T_2^0}\}^2 \\
&+ (1 - \bar{\delta})^2 \sum_{t=T_2^0+2}^{T_{2,\epsilon}^0} y_{t-1}^2 + 2(1 - \bar{\delta}) \sum_{t=T_2^0+2}^{T_{2,\epsilon}^0} y_{t-1} u_t. \quad \text{[using (A.7)]} \qquad \text{(A.8)}
\end{aligned}
$$

Then, subtracting (A.6) from (A.8), we have

$$
F(k_1, 0) = \left[(\delta - \bar{\delta})^2 - (\delta - 1)^2\right] \sum_{t=T_2^0+k_1+1}^{T_2^0} y_{t-1}^2 + \bar{\delta}^2 y_{T_2^0}^2 + s.o.,
$$

and

$$
\begin{aligned}
S_T^{-1} F(k_1, 0) &= \left[(2\delta - \bar{\delta} - 1)(1 - \bar{\delta})(\delta^2 - 1)^{-1}(1 - \delta^{-2|k_1|}) + \bar{\delta}^2\right] S_T^{-1} y_{T_2^0}^2 + o_p(1) \\
&> \left[(\delta^2 - 1)^{-1}(1 - \delta^{-2|k_1|}) + \bar{\delta}^2\right] S_T^{-1} y_{T_2^0}^2 + o_p(1) = O_p^+(1).
\end{aligned}
$$

Case 2: $k_1 = 0,\ k_2 > 0$. Following the same steps used to show (A.4), we can show that

$$
S_T^{1/2} T^{-1/2} \hat{\delta}(T_2^0, T_{2,\epsilon}^0 + k_2) = O_p(1). \qquad \text{(A.9)}
$$

Next, we have

$$SSR(T_2^0, T_{2,\epsilon}^0 + k_2) = \sum_{t=2}^{T_2^0} (\Delta y_t)^2 + \sum_{t=T_2^0+1}^{T_{2,\epsilon}^0+k_2} \{y_t - \hat{\delta}(T_2^0, T_{2,\epsilon}^0 + k_2) y_{t-1}\}^2 + \sum_{t=T_{2,\epsilon}^0+k_2+1}^{T} (\Delta y_t)^2$$

$$= \sum_{t=2}^{T} u_t^2 1(t \neq T_2^0 + 1) + (\delta - 1)^2 \sum_{t=T_1^0+1}^{T_2^0} y_{t-1}^2 + 2(\delta - 1) \sum_{t=T_1^0+1}^{T_2^0} y_{t-1} u_t$$

$$+ \{y_{T_2^0+1} - \hat{\delta}(T_2^0, T_{2,\epsilon}^0 + k_2) y_{T_2^0}\}^2 + (1 - \hat{\delta}(T_2^0, T_{2,\epsilon}^0 + k_2))^2 \sum_{t=T_2^0+2}^{T_{2,\epsilon}^0+k_2} y_{t-1}^2$$

$$+ 2(1 - \hat{\delta}(T_2^0, T_{2,\epsilon}^0 + k_2)) \sum_{t=T_2^0+2}^{T_{2,\epsilon}^0+k_2} y_{t-1} u_t$$

$$= \sum_{t=2}^{T} u_t^2 1(t \neq T_2^0 + 1) + (\delta - 1)^2 \sum_{t=T_1^0+1}^{T_2^0} y_{t-1}^2 + 2(\delta - 1) \sum_{t=T_1^0+1}^{T_2^0} y_{t-1} u_t$$

$$+ \sum_{t=T_2^0+2}^{T_{2,\epsilon}^0+k_2} y_{t-1}^2 + 2 \sum_{t=T_2^0+2}^{T_{2,\epsilon}^0+k_2} y_{t-1} u_t + o_p(1). \quad \text{[using (A.9)]} \tag{A.10}$$

Then, subtracting (A.6) from (A.10), we have

$$F(0, k_2) = \sum_{t=T_{2,\epsilon}^0+1}^{T_{2,\epsilon}^0+k_2} y_{t-1}^2 + 2 \sum_{t=T_{2,\epsilon}^0+1}^{T_{2,\epsilon}^0+k_2} y_{t-1} u_t + s.o.,$$

and

$$T^{-1} F(0, k_2) = T^{-1} y_{T_2^0,\epsilon}^2 k_2 + o_p(1) = O_p^+(1).$$

Combining Cases 1 and 2, it follows that if either k_1 or k_2 is non zero, then $F(k_1, k_2) > 0$ in the limit. Thus, it must be the case that $k_1 = k_2 = 0$ which proves the result. ▲

(b) Using (a), we can write

$$\hat{\delta} = \hat{\delta}(\hat{T}_1, \hat{T}_2) = \hat{\delta}(T_2^0, T_{2,\epsilon}^0) + o_p(1)$$

$$= \left(\sum_{t=T_2^0+1}^{T_{2,\epsilon}^0} y_{t-1}^2 \right)^{-1} \sum_{t=T_2^0+1}^{T_{2,\epsilon}^0} y_t y_{t-1} + o_p(1)$$

$$= \left(y_{T_2^0}^2 + \sum_{t=T_2^0+2}^{T_{2,\epsilon}^0} y_{t-1}^2 \right)^{-1} \left(y_{T_2^0+1} y_{T_2^0} + \sum_{t=T_2^0+2}^{T_{2,\epsilon}^0} y_{t-1}^2 + \sum_{t=T_2^0+2}^{T_{2,\epsilon}^0} y_{t-1} u_t \right) + o_p(1)$$

$$= \left(S_T^{-1} y_{T_2^0}^2 + o_p(1) \right)^{-1} \left(T^{1/2} S_T^{-1/2} T^{-1/2} y_{T_2^0+1} S_T^{-1/2} y_{T_2^0} \right) + o_p(1) \quad \text{[using Lemmas A.1-A.2]}$$

$$= O_p(1).O_p(T^{1/2} S_T^{-1/2}) = o_p(1),$$

which proves the result. ▲

Proof of Theorem 2: (a) Given (T_1, T_2), the sum of squared residuals omitting the residual at time period T_2+1 is given by

$$SSR_{om}(T_1,T_2) = \sum_{t=2}^{T_1} (\Delta y_t)^2 + \sum_{t=T_1+1}^{T_2} \{y_t - \hat{\delta}(T_1,T_2) y_{t-1}\}^2 + \sum_{t=T_2+2}^{T} (\Delta y_t)^2,$$

where $\hat{\delta}(T_1,T_2)$ is as defined in (A.3). Note that

$$\hat{\delta}(T_1^0,T_2^0) = \delta + \left(\sum_{t=T_1^0+1}^{T_2^0} y_{t-1}^2 \right)^{-1} \sum_{t=T_1^0+1}^{T_2^0} y_{t-1} u_t = \delta + O_p(S_T^{-1/2}). \quad \text{[using Lemma A.1]} \tag{A.11}$$

Next, suppose that

$$\hat{T}_1 = T_1^0 + k_1,$$
$$\hat{T}_2 = T_2^0 + k_2,$$

where k_1, k_2 are $O(1)$ integers. Define

$$\begin{aligned} G(k_1,k_2) &= SSR_{om}(\hat{T}_1,\hat{T}_2) - SSR_{om}(T_1^0,T_2^0) \\ &= \left[\sum_{t=2}^{\hat{T}_1} (\Delta y_t)^2 + \sum_{t=\hat{T}_1+1}^{\hat{T}_2} \{y_t - \hat{\delta}(\hat{T}_1,\hat{T}_2) y_{t-1}\}^2 + \sum_{t=\hat{T}_2+2}^{T} (\Delta y_t)^2 \right] \\ &\quad - \left[\sum_{t=2}^{T_1^0} (\Delta y_t)^2 + \sum_{t=T_1^0+1}^{T_2^0} \{y_t - \hat{\delta}(T_1^0,T_2^0) y_{t-1}\}^2 + \sum_{t=T_2^0+2}^{T} (\Delta y_t)^2 \right]. \end{aligned} \tag{A.12}$$

We can write

$$\begin{aligned} SSR_{om}(T_1^0,T_2^0) &= \sum_{t=2}^{T_1^0} (\Delta y_t)^2 + \sum_{t=T_1^0+1}^{T_2^0} \{y_t - \hat{\delta}(T_1^0,T_2^0) y_{t-1}\}^2 + \sum_{t=T_2^0+2}^{T} (\Delta y_t)^2 \\ &= \sum_{t=2}^{T} u_t^2 1(t \neq T_2^0+1) + \left[\delta - \hat{\delta}(T_1^0,T_2^0)\right]^2 \sum_{t=T_1^0+1}^{T_2^0} y_{t-1}^2 + 2\left[\delta - \hat{\delta}(T_1^0,T_2^0)\right] \sum_{t=T_1^0+1}^{T_2^0} y_{t-1} u_t \\ &= \sum_{t=2}^{T} u_t^2 1(t \neq T_2^0+1) + O_p(S_T^{-1}) O_p(S_T) + O_p(S_T^{-1/2}) O_p(S_T^{1/2}) \\ &= \sum_{t=2}^{T} u_t^2 1(t \neq T_2^0+1) + O_p(1). \end{aligned} \tag{A.13}$$

We will show that if either $k_1 \neq 0$ or $k_2 \neq 0$, then $G(k_1,k_2) > 0$ asymptotically. We have four possible cases depending on the signs of k_1 and k_2. We consider each of these in turn.

Case 1: $k_1 > 0,\ k_2 = 0$. First, observe that

$$\hat{\delta}(T_1^0 + k_1, T_2^0) = \delta + O_p(S_T^{-1/2}). \tag{A.14}$$

Next, we can write

$$
\begin{aligned}
SSR_{om}(T_1^0+k_1,T_2^0) &= \sum_{t=2}^{T_1^0+k_1}(\Delta y_t)^2 + \sum_{t=T_1^0+k_1+1}^{T_2^0}\{y_t-\hat{\delta}(T_1^0+k_1,T_2^0)y_{t-1}\}^2 + \sum_{t=T_2^0+2}^{T}(\Delta y_t)^2 \\
&= \sum_{t=2}^{T} u_t^2 1(t\neq T_2^0+1) + (\delta-1)^2\sum_{t=T_1^0+1}^{T_1^0+k_1} y_{t-1}^2 + 2(\delta-1)\sum_{t=T_1^0+1}^{T_1^0+k_1} y_{t-1}u_t \\
&+ \left[\delta-\hat{\delta}(T_1^0+k_1,T_2^0)\right]^2 \sum_{t=T_1^0+k_1+1}^{T_2^0} y_{t-1}^2 + 2\left[\delta-\hat{\delta}(T_1^0+k_1,T_2^0)\right]\sum_{t=T_1^0+k_1+1}^{T_2^0} y_{t-1}u_t \\
&= \sum_{t=2}^{T} u_t^2 1(t\neq T_2^0+1) + (\delta-1)^2\sum_{t=T_1^0+1}^{T_1^0+k_1} y_{t-1}^2 + 2(\delta-1)\sum_{t=T_1^0+1}^{T_1^0+k_1} y_{t-1}u_t + O_p(1) \\
&= \sum_{t=2}^{T} u_t^2 1(t\neq T_2^0+1) + (\delta-1)^2(\delta^2-1)^{-1}\left(\delta^{2k_1}-1\right) y_{T_1^0}^2 \\
&+ 2(\delta-1)(\sum_{t=T_1^0+1}^{T_1^0+k_1}\delta^{t-T_1^0-1}u_t)y_{T_1^0} + O_p(1). \qquad \text{(A.15)}
\end{aligned}
$$

Now, using (A.14), (A.15) simplifies to

$$
\begin{aligned}
&= \sum_{t=2}^{T} u_t^2 1(t\neq T_2^0+1) + (\delta-1)^2\sum_{t=T_1^0+1}^{T_1^0+k_1} y_{t-1}^2 + 2(\delta-1)\sum_{t=T_1^0+1}^{T_1^0+k_1} y_{t-1}u_t + O_p(1) \\
&= \sum_{t=2}^{T} u_t^2 1(t\neq T_2^0+1) + (\delta-1)^2(\delta^2-1)^{-1}\left(\delta^{2k_1}-1\right) y_{T_1^0}^2 \\
&+ 2(\delta-1)(\sum_{t=T_1^0+1}^{T_1^0+k_1}\delta^{t-T_1^0-1}u_t)y_{T_1^0} + O_p(1), \qquad \text{(A.16)}
\end{aligned}
$$

where the second equality uses (A.14). Then, subtracting (A.13) from (A.16), we have

$$
G(k_1,0) = (\delta-1)^2(\delta^2-1)^{-1}\left(\delta^{2k_1}-1\right) y_{T_1^0}^2 + 2(\delta-1)(\sum_{t=T_1^0+1}^{T_1^0+k_1}\delta^{t-T_1^0-1}u_t)y_{T_1^0} + O_p(1). \qquad \text{(A.17)}
$$

Since $E\left(\sum_{t=T_1^0+1}^{T_1^0+k_1}\delta^{t-T_1^0-1}u_t\right)^2 = \sigma^2\sum_{s=0}^{k_1-1}\delta^{2s} = O(1)$, the second term in (A.17) is $O_p(T^{1/2})$. Then, using the fact that $T^{-1/2}y_{T_1^0} = O_p(1)$, we get

$$
T^{-1}G(k_1,0) = (\delta-1)^2(\delta^2-1)^{-1}\left(\delta^{2k_1}-1\right) T^{-1}y_{T_1^0}^2 + o_p(1) = O_p^{+}(1).
$$

Case 2: $k_1 < 0,\ k_2 = 0$. Again, note that $\hat{\delta}(T_1^0 + k_1, T_2^0) = \delta + O_p(S_T^{-1/2})$. Next, we can write

$$SSR_{om}(T_1^0 + k_1, T_2^0) = \sum_{t=2}^{T} u_t^2 1(t \neq T_2^0 + 1) + \left[1 - \hat{\delta}(T_1^0 + k_1, T_2^0)\right]^2 \sum_{t=T_1^0+k_1+1}^{T_1^0} y_{t-1}^2$$
$$+ 2\left[1 - \hat{\delta}(T_1^0 + k_1, T_2^0)\right] \sum_{t=T_1^0+k_1+1}^{T_1^0} y_{t-1}u_t + \left[\delta - \hat{\delta}(T_1^0 + k_1, T_2^0)\right]^2 \sum_{t=T_1^0+1}^{T_2^0} y_{t-1}^2$$
$$+ 2\left[\delta - \hat{\delta}(T_1^0 + k_1, T_2^0)\right] \sum_{t=T_1^0+1}^{T_2^0} y_{t-1}u_t. \tag{A.18}$$

Then, subtracting (A.13) from (A.18) and scaling the difference by T^{-1}, we have

$$T^{-1}G(k_1, 0) = (\delta - 1)^2 T^{-1} \sum_{t=T_1^0+k_1+1}^{T_1^0} y_{t-1}^2 + 2(1-\delta) \sum_{t=T_1^0+k_1+1}^{T_1^0} y_{t-1}u_t + o_p(1)$$
$$= (\delta - 1)^2 |k_1| T^{-1} y_{T_1^0}^2 + 2(1-\delta) T^{-1/2} y_{T_1^0} T^{-1/2} \sum_{t=T_1^0+k_1+1}^{T_1^0} u_t + o_p(1)$$
$$= (\delta - 1)^2 |k_1| T^{-1} y_{T_1^0}^2 + o_p(1) = O_p^+(1).$$

Case 3: $k_1 = 0,\ k_2 > 0$. First, observe that

$$\hat{\delta}(T_1^0, T_2^0 + k_2) = \left(\sum_{t=T_1^0+1}^{T_2^0+k_2} y_{t-1}^2\right)^{-1} \sum_{t=T_1^0+1}^{T_2^0+k_2} y_t y_{t-1}$$
$$= \delta + \left(S_T^{-1} \sum_{t=T_1^0+1}^{T_2^0} y_{t-1}^2 + S_T^{-1} y_{T_2^0}^2 + o_p(1)\right)^{-1} \left(-\delta S_T^{-1} y_{T_2^0}^2 + S_T^{-1} y_{T_2^0+1} y_{T_2^0}\right) + o_p(1)$$
$$= \delta - \delta[(\delta^2 - 1)^{-1} + 1]^{-1} + O_p(T^{1/2} S_T^{-1/2})$$
$$= \delta^{-1} + o_p(1). \tag{A.19}$$

Next, we have

$$SSR_{om}(T_1^0, T_2^0 + k_2) = \sum_{t=2}^{T} u_t^2 1(t \neq T_2^0 + 1) + \left[\delta - \hat{\delta}(T_1^0, T_2^0 + k_2)\right]^2 \sum_{t=T_1^0+1}^{T_2^0} y_{t-1}^2$$
$$+ 2\left[\delta - \hat{\delta}(T_1^0, T_2^0 + k_2)\right] \sum_{t=T_1^0+1}^{T_2^0} y_{t-1}u_t + \{y_{T_2^0+1} - \hat{\delta}(T_1^0, T_2^0 + k_2) y_{T_2^0}\}^2$$
$$+ [1 - \hat{\delta}(T_1^0, T_2^0 + k_2)]^2 \sum_{t=T_2^0+2}^{T_2^0+k_2} y_{t-1}^2 + 2[1 - \hat{\delta}(T_1^0, T_2^0 + k_2)] \sum_{t=T_2^0+2}^{T_2^0+k_2} y_{t-1}u_t. \tag{A.20}$$

Then, subtracting (A.13) from (A.20) and scaling the difference by S_T^{-1}, we have

$$\begin{aligned}
S_T^{-1}G(0,k_2) &= (\delta-\delta^{-1})^2 S_T^{-1}\sum_{t=T_1^0+1}^{T_2^0} y_{t-1}^2 + 2(\delta-\delta^{-1})S_T^{-1}\sum_{t=T_1^0+1}^{T_2^0} y_{t-1}u_t + \delta^{-2}S_T^{-1}y_{T_2^0}^2 \\
&+ (1-\delta^{-1})^2 S_{T^-}^{-1}\sum_{t=T_2^0+2}^{T_2^0+k_2} y_{t-1}^2 + 2(1-\delta^{-1})S_T^{-1}\sum_{t=T_2^0+2}^{T_2^0+k_2} y_{t-1}u_t + o_p(1) \\
&= (\delta-\delta^{-1})^2 S_T^{-1}\sum_{t=T_1^0+1}^{T_2^0} y_{t-1}^2 + \delta^{-2}S_T^{-1}y_{T_2^0}^2 + o_p(1) \\
&= [(\delta-\delta^{-1})^2(\delta^2-1)^{-1}+\delta^{-2}]S_T^{-1}y_{T_2^0}^2 + o_p(1) = S_T^{-1}y_{T_2^0}^2 + o_p(1) = O_p^+(1),
\end{aligned}$$

where the first equality uses (A.19) and the second and third use Lemma A.1.

Case 4: $k_1 = 0,\ k_2 < 0$. Observe that in this case,

$$\hat{\delta}(T_1^0, T_2^0+k_2) = \delta + O_p(S_T^{-1/2}). \tag{A.21}$$

Next, we can write

$$\begin{aligned}
SSR_{om}(T_1^0, T_2^0+k_2) &= \sum_{t=2}^{T} u_t^2 1(t \neq T_2^0+1) + \left[\delta - \hat{\delta}(T_1^0, T_2^0+k_2)\right]^2 \sum_{t=T_1^0+1}^{T_2^0+k_2} y_{t-1}^2 \\
&+ 2\left[\delta - \hat{\delta}(T_1^0+k_1, T_2^0)\right]\sum_{t=T_1^0+1}^{T_2^0+k_2} y_{t-1}u_t + (\delta-1)^2 \sum_{t=T_2^0+k_2+2}^{T_2^0} y_{t-1}^2 \\
&+ 2(\delta-1)\sum_{t=T_2^0+k_2+2}^{T_2^0} y_{t-1}u_t + (y_{T_2^0+1} - y_{T_2^0})^2.
\end{aligned} \tag{A.22}$$

Then, subtracting (A.13) from (A.22), scaling the difference by S_T^{-1}, and using (A.21), we have

$$\begin{aligned}
S_T^{-1}G(0,k_2) &= (\delta-1)^2 S_T^{-1}\sum_{t=T_2^0+k_2+2}^{T_2^0} y_{t-1}^2 + S_T^{-1}y_{T_2^0}^2 + o_p(1) \\
&= \left[(\delta-1)^2(\delta^2-1)^{-1}(1-\delta^{-2(|k_2|-1)})+1\right] S_T^{-1}y_{T_2^0}^2 + o_p(1) = O_p^+(1).
\end{aligned}$$

Combining cases 1-4, it follows that if either k_1 or k_2 is non zero, then $G(k_1, k_2) > 0$ in the limit. Thus, it must be the case that $k_1 = k_2 = 0$ which proves the result. ▲

(b) Using (a), we can write

$$
\begin{aligned}
\hat{\delta} &= \hat{\delta}(\hat{T}_1, \hat{T}_2) = \hat{\delta}(T_1^0, T_2^0) + o_p(1) \\
&= \left(\sum_{t=T_1^0+1}^{T_2^0} y_{t-1}^2 \right)^{-1} \sum_{t=T_1^0+1}^{T_2^0} y_t y_{t-1} + o_p(1) \\
&= \delta + \left(\sum_{t=T_1^0+1}^{T_2^0} y_{t-1}^2 \right)^{-1} \sum_{t=T_1^0+1}^{T_2^0} y_{t-1} u_t + o_p(1) \\
&= \delta + [O_p(S_T)]^{-1} O_p(S_T^{-1/2}) + o_p(1) \quad \text{[using Lemma A.1]} \\
&= \delta + O_p(S_T^{-1/2}) \xrightarrow{p} \delta,
\end{aligned}
$$

thereby proving the result. ▲

References

Harvey DI, Leybourne SJ, Sollis R. 2017. Improving the accuracy of asset price bubble start and end date estimators. *Journal of Empirical Finance* **40**:121–138.

Perron P. 1989. The great crash, the oil price shock, and the unit root hypothesis. *Econometrica: journal of the Econometric Society* **57**(6):1361–1401.

Perron P. 1990. Testing for a unit root in a time series with a changing mean. *Journal of Business & Economic Statistics* **8**(2):153–162.

Appendix B: Additional Monte Carlo Results

Table B.1: Probabilities of break date selection (single bubble; $u_t = 0.5u_{t-1} + e_t$).

Panel A: First date estimates

δ	T	T_1^0	$\widehat{p}_1^C(JS)$	$\widehat{p}_1^L(JS)$	$\widehat{p}_1^C(PQ)$	$\widehat{p}_1^L(PQ)$	$\widetilde{p}_1^C(JS)$	$\widetilde{p}_1^L(JS)$	$\widetilde{p}_1^C(PQ)$	$\widetilde{p}_1^L(PQ)$	$\widehat{p}_1^C(PSY)$	$\widehat{p}_1^L(PSY)$
1.02	100	50	0.00	0.80	0.00	0.72	**0.03**	0.25	0.02	0.35	0.02	0.57
1.02	200	100	0.00	0.89	0.00	0.85	**0.03**	0.32	0.02	0.41	0.01	0.69
1.02	400	200	0.00	0.94	0.00	0.92	**0.03**	0.45	0.02	0.45	0.00	0.82
1.02	100	40	0.00	0.90	0.00	0.85	**0.03**	0.36	0.02	0.51	0.02	0.73
1.02	200	80	0.00	0.94	0.00	0.92	**0.03**	0.44	0.01	0.55	0.01	0.81
1.02	400	160	0.00	0.97	0.00	0.96	**0.03**	0.54	0.01	0.56	0.00	0.89
1.05	100	50	0.00	0.94	0.00	0.90	**0.09**	0.25	0.05	0.33	0.03	0.74
1.05	200	100	0.00	0.98	0.00	0.97	**0.13**	0.36	0.07	0.34	0.01	0.88
1.05	400	200	0.00	0.99	0.00	0.99	**0.16**	0.43	0.10	0.34	0.01	0.94
1.05	100	40	0.00	0.98	0.00	0.96	**0.07**	0.35	0.03	0.48	0.02	0.86
1.05	200	80	0.00	0.99	0.00	0.99	**0.10**	0.44	0.05	0.48	0.01	0.93
1.05	400	160	0.00	1.00	0.00	1.00	**0.15**	0.47	0.09	0.39	0.01	0.95

Panel B: Second date estimates

δ	T	T_2^0	$\widehat{p}_2^C(JS)$	$\widehat{p}_2^L(JS)$	$\widehat{p}_2^C(PQ)$	$\widehat{p}_2^L(PQ)$	$\widetilde{p}_2^C(JS)$	$\widetilde{p}_2^L(JS)$	$\widetilde{p}_2^C(PQ)$	$\widetilde{p}_2^L(PQ)$	$\widehat{p}_2^C(PSY)$	$\widehat{p}_2^L(PSY)$
1.02	100	65	0.00	0.86	0.14	0.78	**0.77**	0.15	0.58	0.35	0.20	0.29
1.02	200	130	0.00	0.92	0.08	0.87	**0.90**	0.06	0.67	0.29	0.33	0.27
1.02	400	260	0.00	0.96	0.06	0.92	**0.95**	0.03	0.78	0.20	0.59	0.18
1.02	100	60	0.00	0.93	0.08	0.87	**0.83**	0.13	0.56	0.40	0.23	0.34
1.02	200	120	0.00	0.96	0.05	0.92	**0.92**	0.05	0.63	0.35	0.39	0.28
1.02	400	240	0.00	0.97	0.04	0.95	**0.97**	0.02	0.76	0.23	0.69	0.16
1.05	100	65	0.00	0.96	0.05	0.92	**0.92**	0.06	0.72	0.26	0.58	0.18
1.05	200	130	0.00	0.99	0.02	0.98	**0.98**	0.01	0.87	0.12	0.81	0.10
1.05	400	260	0.00	0.99	0.01	0.99	**0.99**	0.00	0.98	0.02	0.80	0.18
1.05	100	60	0.00	0.98	0.03	0.96	**0.95**	0.04	0.67	0.32	0.66	0.16
1.05	200	120	0.00	1.00	0.01	0.99	**0.99**	0.01	0.83	0.17	0.85	0.11
1.05	400	240	0.00	1.00	0.00	1.00	**1.00**	0.00	0.97	0.03	0.73	0.27

Note: 1) The superscript 'C' denotes the probability of correctly selecting the true break date.
2) The superscript 'L' denotes the probability of selecting a date later than the true break date.
3) The notation "ˆ" indicates "no omission" and "˜" indicates "with omission".
4) The method with the highest probability of correctly selecting the true break date is highlighted in bold.

Table B.2: Bias and RMSE of break fraction estimates (single bubble; $u_t = 0.5u_{t-1} + e_t$).

δ	λ_1^0	λ_2^0	T	$\widehat{\lambda}_1^{JS}$	$\widehat{\lambda}_1^{PQ}$	$\widetilde{\lambda}_1^{JS}$	$\widetilde{\lambda}_1^{PQ}$	$\widehat{\lambda}_1^{PSY}$	$\widehat{\lambda}_2^{JS}$	$\widehat{\lambda}_2^{PQ}$	$\widetilde{\lambda}_2^{JS}$	$\widetilde{\lambda}_2^{PQ}$	$\widehat{\lambda}_2^{PSY}$
						Panel A: Bias							
1.02	0.5	0.65	100	0.075	**0.045**	−0.121	−0.083	0.052	0.075	0.089	**0.011**	0.042	−0.015
1.02	0.5	0.65	200	0.111	0.095	−0.091	**−0.057**	0.067	0.084	0.094	**0.001**	0.033	−0.019
1.02	0.5	0.65	400	0.129	0.120	−0.038	**−0.031**	0.068	0.090	0.095	**0.000**	0.021	−0.019
1.02	0.4	0.6	100	0.164	0.141	−0.053	**0.006**	0.149	0.103	0.115	**0.018**	0.065	0.037
1.02	0.4	0.6	200	0.182	0.170	−0.034	**0.021**	0.144	0.098	0.105	**0.004**	0.047	0.019
1.02	0.4	0.6	400	0.189	0.184	**−0.002**	0.020	0.117	0.096	0.102	**0.002**	0.029	0.007
1.05	0.5	0.65	100	0.130	0.114	−0.090	−0.066	**0.063**	0.101	0.112	**0.005**	0.034	0.005
1.05	0.5	0.65	200	0.144	0.141	**−0.027**	−0.033	0.062	0.100	0.109	**0.001**	0.014	0.001
1.05	0.5	0.65	400	0.147	0.146	**−0.003**	−0.015	0.042	0.099	0.103	**−0.001**	0.002	−0.002
1.05	0.4	0.6	100	0.193	0.184	−0.050	**0.003**	0.118	0.109	0.126	**0.006**	0.049	0.027
1.05	0.4	0.6	200	0.198	0.197	−0.007	**0.006**	0.089	0.103	0.112	**0.001**	0.021	0.010
1.05	0.4	0.6	400	0.199	0.199	0.006	**−0.004**	0.048	0.101	0.104	**0.000**	0.003	0.001
						Panel B: RMSE							
1.02	0.5	0.65	100	**0.182**	0.184	0.209	0.202	0.232	0.157	0.144	**0.099**	0.113	0.225
1.02	0.5	0.65	200	**0.167**	0.168	0.174	0.181	0.227	0.135	0.129	**0.069**	0.095	0.219
1.02	0.5	0.65	400	0.158	0.159	**0.113**	0.140	0.189	0.120	0.115	**0.046**	0.069	0.178
1.02	0.4	0.6	100	0.206	0.197	**0.164**	0.173	0.268	0.148	0.147	**0.091**	0.125	0.225
1.02	0.4	0.6	200	0.204	0.199	**0.134**	0.160	0.250	0.128	0.126	**0.062**	0.099	0.202
1.02	0.4	0.6	400	0.202	0.200	**0.091**	0.131	0.198	0.115	0.114	**0.039**	0.071	0.150
1.05	0.5	0.65	100	**0.159**	0.161	0.165	0.178	0.182	0.127	0.135	**0.061**	0.087	0.168
1.05	0.5	0.65	200	0.153	0.153	**0.082**	0.117	0.133	0.110	0.117	**0.030**	0.052	0.112
1.05	0.5	0.65	400	0.151	0.151	**0.041**	0.060	0.080	0.104	0.107	**0.018**	0.024	0.060
1.05	0.4	0.6	100	0.202	0.199	**0.133**	0.162	0.198	0.123	0.141	**0.051**	0.098	0.150
1.05	0.4	0.6	200	0.200	0.200	**0.072**	0.119	0.140	0.106	0.116	**0.021**	0.055	0.090
1.05	0.4	0.6	400	0.200	0.200	**0.039**	0.061	0.076	0.102	0.105	**0.007**	0.021	0.038

Note: 1) The notation "ˆ" indicates "no omission" and "˜" indicates "with omission".
2) The best method is highlighted in bold.

Table B.3: Bias and RMSE of AR(1) estimates (single bubble; $u_t = 0.5u_{t-1} + e_t$).

δ	λ_1^0	λ_2^0	T	$\widehat{\delta}^{JS}$	$\widehat{\delta}^{PQ}$	$\widetilde{\delta}^{JS}$	$\widetilde{\delta}^{PQ}$	$\widehat{\delta}^{PSY}$
					Panel A: Bias			
1.02	0.5	0.65	100	−0.628	−0.593	**−0.099**	−0.199	−0.184
1.02	0.5	0.65	200	−0.639	−0.622	**−0.060**	−0.159	−0.168
1.02	0.5	0.65	400	−0.664	−0.655	**−0.027**	−0.097	−0.119
1.02	0.4	0.6	100	−0.716	−0.693	**−0.101**	−0.255	−0.183
1.02	0.4	0.6	200	−0.716	−0.703	**−0.059**	−0.221	−0.159
1.02	0.4	0.6	400	−0.756	−0.750	**−0.023**	−0.138	−0.095
1.05	0.5	0.65	100	−0.884	−0.857	**−0.071**	−0.210	−0.137
1.05	0.5	0.65	200	−0.932	−0.926	**−0.025**	−0.101	−0.068
1.05	0.5	0.65	400	−0.991	−0.990	**−0.006**	−0.014	−0.045
1.05	0.4	0.6	100	−0.939	−0.926	**−0.061**	−0.285	−0.117
1.05	0.4	0.6	200	−0.978	−0.974	**−0.019**	−0.160	−0.053
1.05	0.4	0.6	400	−1.026	−1.025	**−0.002**	−0.025	−0.047
					Panel B: RMSE			
1.02	0.5	0.65	100	0.762	0.719	**0.162**	0.365	0.234
1.02	0.5	0.65	200	0.722	0.711	**0.093**	0.316	0.220
1.02	0.5	0.65	400	0.735	0.731	**0.046**	0.243	0.196
1.02	0.4	0.6	100	0.823	0.791	**0.163**	0.435	0.235
1.02	0.4	0.6	200	0.785	0.775	**0.091**	0.402	0.215
1.02	0.4	0.6	400	0.813	0.811	**0.041**	0.320	0.178
1.05	0.5	0.65	100	0.938	0.916	**0.130**	0.414	0.206
1.05	0.5	0.65	200	0.956	0.953	**0.056**	0.284	0.139
1.05	0.5	0.65	400	1.004	1.004	**0.022**	0.094	0.104
1.05	0.4	0.6	100	0.973	0.962	**0.114**	0.504	0.187
1.05	0.4	0.6	200	0.992	0.989	**0.044**	0.385	0.122
1.05	0.4	0.6	400	1.031	1.031	**0.014**	0.156	0.092

Note: 1) The notation "ˆ" indicates "no omission" and "˜" indicates "with omission".
2) The best method is highlighted in bold.

Table B.4: Probabilities of break date selection (two bubbles; $u_t = 0.5u_{t-1} + e_t$).

δ_1	T	T_i^0	$\widehat{p}_i^C(JS)$	$\widehat{p}_i^L(JS)$	$\widehat{p}_i^C(PQ)$	$\widehat{p}_i^L(PQ)$	$\widetilde{p}_i^C(JS)$	$\widetilde{p}_i^L(JS)$	$\widetilde{p}_i^C(PQ)$	$\widetilde{p}_i^L(PQ)$	$\widehat{p}_i^C(PSY)$	$\widehat{p}_i^L(PSY)$	$\widehat{p}_i^C(HLW)$	$\widehat{p}_i^L(HLW)$
Panel A: First break date [i=1]														
1.05	100	20	0.00	0.97	0.01	0.92	**0.07**	0.37	0.05	0.49	0.02	0.91	**0.07**	0.38
1.05	200	40	0.00	0.98	0.00	0.98	**0.08**	0.47	0.04	0.54	0.01	0.94	0.07	0.47
1.05	400	80	0.00	0.99	0.00	1.00	**0.09**	0.51	0.05	0.48	0.01	0.96	**0.09**	0.51
Panel B: Second break date [i=2]														
1.05	100	40	0.00	0.97	0.11	0.89	**0.97**	0.01	0.68	0.27	0.65	0.09	0.92	0.04
1.05	200	80	0.00	0.98	0.03	0.97	**0.99**	0.01	0.73	0.23	0.85	0.09	0.96	0.03
1.05	400	160	0.00	0.99	0.01	0.99	**1.00**	0.00	0.86	0.13	0.74	0.26	0.99	0.01
δ_2	T	T_i^0	$\widehat{p}_i^C(JS)$	$\widehat{p}_i^L(JS)$	$\widehat{p}_i^C(PQ)$	$\widehat{p}_i^L(PQ)$	$\widetilde{p}_i^C(JS)$	$\widetilde{p}_i^L(JS)$	$\widetilde{p}_i^C(PQ)$	$\widetilde{p}_i^L(PQ)$	$\widehat{p}_i^C(PSY)$	$\widehat{p}_i^L(PSY)$	$\widehat{p}_i^C(HLW)$	$\widehat{p}_i^L(HLW)$
Panel C: Third break date [i=3]														
1.05	100	60	0.00	0.97	0.02	0.90	**0.12**	0.39	0.09	0.52	0.02	0.87	**0.12**	0.40
1.05	200	120	0.00	0.98	0.00	0.98	**0.13**	0.45	0.11	0.50	0.01	0.89	0.12	0.46
1.05	400	240	0.00	0.99	0.00	1.00	**0.16**	0.45	0.12	0.43	0.01	0.92	**0.16**	0.44
Panel D: Fourth break date [i=4]														
1.05	100	80	0.00	0.97	0.15	0.84	**0.97**	0.02	0.77	0.23	0.82	0.02	0.93	0.04
1.05	200	160	0.00	0.98	0.04	0.96	**0.99**	0.01	0.86	0.14	0.88	0.07	0.96	0.03
1.05	400	320	0.00	0.99	0.01	0.99	**1.00**	0.00	0.93	0.07	0.73	0.26	0.99	0.01

Note: See notes to Table B.1.

Table B.5: Bias and RMSE of break fraction estimates (two bubbles; $u_t = 0.5u_{t-1} + e_t$).

δ_1	λ_1^0	λ_2^0	T	$\widehat{\lambda}_1^{JS}$	$\widehat{\lambda}_2^{JS}$	$\widehat{\lambda}_1^{PQ}$	$\widehat{\lambda}_2^{PQ}$	$\widetilde{\lambda}_1^{JS}$	$\widetilde{\lambda}_2^{JS}$	$\widetilde{\lambda}_1^{PQ}$	$\widetilde{\lambda}_2^{PQ}$	$\widehat{\lambda}_1^{PSY}$	$\widehat{\lambda}_2^{PSY}$	$\widehat{\lambda}_1^{HLW}$	$\widehat{\lambda}_2^{HLW}$
Panel A: Bias of first bubble estimators															
1.05	0.2	0.4	100	0.194	0.110	0.175	0.118	−0.014	0.000^c	0.039	0.032	0.102	0.002	-0.008^s	0.001
1.05	0.2	0.4	200	0.199	0.104	0.194	0.111	0.001^s	0.001^c	0.039	0.022	0.096	0.013	0.007	0.007
1.05	0.2	0.4	400	0.200	0.102	0.199	0.104	0.009^s	0.000^c	0.021	0.012	0.055	0.003	0.011	0.003
Panel B: RMSE of first bubble estimators															
1.05	0.2	0.4	100	0.200	0.125	0.189	0.134	0.064^s	0.021^c	0.116	0.082	0.141	0.085	0.075	0.047
1.05	0.2	0.4	200	0.203	0.113	0.198	0.117	0.057^s	0.010^c	0.108	0.064	0.136	0.081	0.078	0.060
1.05	0.2	0.4	400	0.202	0.105	0.200	0.106	0.040^s	0.003^c	0.082	0.041	0.078	0.036	0.052	0.034
δ_2	T	λ_3^0	λ_4^0	$\widehat{\lambda}_3^{JS}$	$\widehat{\lambda}_4^{JS}$	$\widehat{\lambda}_3^{PQ}$	$\widehat{\lambda}_4^{PQ}$	$\widetilde{\lambda}_3^{JS}$	$\widetilde{\lambda}_4^{JS}$	$\widetilde{\lambda}_3^{PQ}$	$\widetilde{\lambda}_4^{PQ}$	$\widehat{\lambda}_3^{PSY}$	$\widehat{\lambda}_4^{PSY}$	$\widehat{\lambda}_3^{HLW}$	$\widehat{\lambda}_4^{HLW}$
Panel C: Bias of second bubble estimators															
1.05	0.6	0.8	100	0.187	0.087	0.163	0.082	−0.010	0.001^c	0.029	0.021	0.063	−0.022	-0.009^s	−0.005
1.05	0.6	0.8	200	0.193	0.093	0.190	0.095	0.003^s	0.000^c	0.021	0.014	0.062	−0.006	0.006	0.001
1.05	0.6	0.8	400	0.197	0.097	0.199	0.099	0.006^s	0.000^c	0.010	0.006	0.039	0.000^c	0.007	0.001
Panel D: RMSE of second bubble estimators															
1.05	0.6	0.8	100	0.199	0.111	0.184	0.092	0.064^s	0.017^c	0.112	0.049	0.109	0.091	0.075	0.047
1.05	0.6	0.8	200	0.200	0.106	0.196	0.099	0.053^s	0.009^c	0.094	0.039	0.108	0.073	0.078	0.060
1.05	0.6	0.8	400	0.200	0.103	0.199	0.100	0.036^s	0.003^c	0.065	0.026	0.065	0.029	0.052	0.034

Note: 1) The superscript 's' denotes bubble origination estimates with lowest bias/RMSE.
2) The superscript 'c' denotes bubble crash estimates with lowest bias/RMSE.
3) The notation "ˆ" indicates "no omission" and "˜" indicates "with omission".

Table B.6: Bias and RMSE of AR(1) estimates (two bubbles; $u_t = 0.5u_{t-1} + e_t$).

δ	T	$\widehat{\delta}_1^{JS}$	$\widehat{\delta}_1^{PQ}$	$\widetilde{\delta}_1^{JS}$	$\widetilde{\delta}_1^{PQ}$	$\widehat{\delta}_1^{PSY}$	$\widehat{\delta}_1^{HLW}$	$\widehat{\delta}_2^{JS}$	$\widehat{\delta}_2^{PQ}$	$\widetilde{\delta}_2^{JS}$	$\widetilde{\delta}_2^{PQ}$	$\widehat{\delta}_2^{PSY}$	$\widehat{\delta}_2^{HLW}$
Panel A: Bias													
1.05	100	−0.940	−0.870	**−0.060**	−0.300	−0.300	**−0.060**	−1.000	−0.850	**−0.060**	−0.250	−0.090	−0.100
1.05	200	−0.950	−0.940	**−0.020**	−0.230	−0.230	**−0.020**	−1.000	−0.960	**−0.020**	−0.140	−0.040	−0.030
1.05	400	−1.010	−1.020	**0.000**	−0.130	−0.130	**0.000**	−1.030	−1.030	**0.000**	−0.070	−0.050	**0.000**
Panel B: RMSE													
1.05	100	0.980	0.940	**0.120**	0.520	0.520	**0.120**	1.010	0.930	**0.100**	0.470	0.170	0.240
1.05	200	0.980	0.970	**0.050**	0.470	0.470	**0.050**	1.000	0.980	**0.050**	0.360	0.110	0.120
1.05	400	1.020	1.030	**0.010**	0.360	0.360	**0.010**	1.040	1.030	**0.010**	0.260	0.090	0.040

Note: See notes to Table B.3.

Table B.7: Probabilities of break date selection (single bubble; $u_t = e_t + 0.5e_{t-1}$).

Panel A: First date estimates												
δ	T	T_1^0	$\widehat{p}_1^C(JS)$	$\widehat{p}_1^L(JS)$	$\widehat{p}_1^C(PQ)$	$\widehat{p}_1^L(PQ)$	$\widetilde{p}_1^C(JS)$	$\widetilde{p}_1^L(JS)$	$\widetilde{p}_1^C(PQ)$	$\widetilde{p}_1^L(PQ)$	$\widehat{p}_1^C(PSY)$	$\widehat{p}_1^L(PSY)$
1.02	100	50	0.00	0.81	0.00	0.73	**0.03**	0.24	0.02	0.34	0.02	0.59
1.02	200	100	0.00	0.93	0.00	0.90	**0.03**	0.32	0.02	0.40	0.01	0.77
1.02	400	200	0.00	0.97	0.00	0.96	**0.04**	0.46	0.02	0.41	0.00	0.90
1.02	100	40	0.00	0.91	0.00	0.86	**0.03**	0.37	0.01	0.50	0.02	0.76
1.02	200	80	0.00	0.97	0.00	0.95	**0.03**	0.44	0.01	0.54	0.01	0.87
1.02	400	160	0.00	0.99	0.00	0.98	**0.03**	0.56	0.01	0.53	0.00	0.95
1.05	100	50	0.00	0.96	0.00	0.93	**0.09**	0.26	0.05	0.33	0.02	0.83
1.05	200	100	0.00	0.99	0.00	0.99	**0.12**	0.40	0.06	0.34	0.01	0.96
1.05	400	200	0.00	1.00	0.00	1.00	**0.15**	0.47	0.09	0.35	0.00	0.97
1.05	100	40	0.00	0.99	0.00	0.98	**0.08**	0.37	0.04	0.47	0.01	0.92
1.05	200	80	0.00	1.00	0.00	1.00	**0.09**	0.48	0.04	0.45	0.01	0.97
1.05	400	160	0.00	1.00	0.00	1.00	**0.13**	0.52	0.07	0.40	0.00	0.98
Panel B: Second date estimates												
δ	T	T_2^0	$\widehat{p}_2^C(JS)$	$\widehat{p}_2^L(JS)$	$\widehat{p}_2^C(PQ)$	$\widehat{p}_2^L(PQ)$	$\widetilde{p}_2^C(JS)$	$\widetilde{p}_2^L(JS)$	$\widetilde{p}_2^C(PQ)$	$\widetilde{p}_2^L(PQ)$	$\widehat{p}_2^C(PSY)$	$\widehat{p}_2^L(PSY)$
1.02	100	65	0.00	0.89	0.13	0.79	**0.78**	0.15	0.58	0.35	0.23	0.24
1.02	200	130	0.00	0.96	0.06	0.91	**0.93**	0.04	0.71	0.26	0.45	0.17
1.02	400	260	0.00	0.98	0.03	0.96	**0.98**	0.01	0.84	0.15	0.75	0.10
1.02	100	60	0.00	0.94	0.07	0.89	**0.83**	0.13	0.56	0.41	0.24	0.27
1.02	200	120	0.00	0.98	0.03	0.95	**0.95**	0.04	0.67	0.32	0.52	0.18
1.02	400	240	0.00	0.99	0.01	0.98	**0.99**	0.01	0.80	0.20	0.83	0.08
1.05	100	65	0.00	0.98	0.04	0.94	**0.94**	0.04	0.75	0.23	0.70	0.11
1.05	200	130	0.00	1.00	0.01	0.99	**0.99**	0.00	0.91	0.09	0.91	0.04
1.05	400	260	0.00	1.00	0.00	1.00	**1.00**	0.00	0.99	0.01	0.87	0.12
1.05	100	60	0.00	0.99	0.02	0.98	**0.97**	0.02	0.71	0.28	0.76	0.09
1.05	200	120	0.00	1.00	0.00	1.00	**1.00**	0.00	0.85	0.15	0.92	0.05
1.05	400	240	0.00	1.00	0.00	1.00	**1.00**	0.00	0.98	0.02	0.78	0.22

Note: See notes to Table B.1.

Table B.8: Bias and RMSE of break fraction estimates (single bubble; $u_t = e_t + 0.5e_{t-1}$).

δ	λ_1^0	λ_2^0	T	$\widehat{\lambda}_1^{JS}$	$\widehat{\lambda}_1^{PQ}$	$\widetilde{\lambda}_1^{JS}$	$\widetilde{\lambda}_1^{PQ}$	$\widehat{\lambda}_1^{PSY}$	$\widehat{\lambda}_2^{JS}$	$\widehat{\lambda}_2^{PQ}$	$\widetilde{\lambda}_2^{JS}$	$\widetilde{\lambda}_2^{PQ}$	$\widehat{\lambda}_2^{PSY}$
							Panel A: Bias						
1.02	0.5	0.65	100	0.081	0.047	−0.120	−0.088	**0.045**	0.091	0.094	**0.011**	0.043	−0.028
1.02	0.5	0.65	200	0.125	0.114	−0.086	**−0.058**	0.063	0.097	0.104	**0.002**	0.032	−0.025
1.02	0.5	0.65	400	0.139	0.135	**−0.027**	−0.038	0.074	0.097	0.102	**−0.001**	0.016	−0.011
1.02	0.4	0.6	100	0.168	0.148	−0.050	**0.007**	0.136	0.112	0.124	0.018	0.066	**0.016**
1.02	0.4	0.6	200	0.189	0.182	−0.034	**0.014**	0.134	0.103	0.114	**0.004**	0.044	0.005
1.02	0.4	0.6	400	0.196	0.193	**0.000**	0.010	0.107	0.101	0.106	**0.001**	0.023	0.001
1.05	0.5	0.65	100	0.138	0.126	−0.080	−0.069	**0.062**	0.110	0.124	0.004	0.033	**−0.002**
1.05	0.5	0.65	200	0.148	0.146	**−0.017**	−0.038	0.068	0.104	0.116	**0.000**	0.011	0.003
1.05	0.5	0.65	400	0.149	0.149	**0.001**	−0.016	0.045	0.101	0.106	**−0.000**	0.001	−0.001
1.05	0.4	0.6	100	0.196	0.191	−0.043	**0.001**	0.109	0.113	0.135	**0.004**	0.046	0.011
1.05	0.4	0.6	200	0.199	0.199	**−0.002**	−0.003	0.089	0.105	0.117	**0.001**	0.019	0.005
1.05	0.4	0.6	400	0.200	0.200	0.010	**−0.008**	0.052	0.102	0.107	**0.000**	0.002	0.001
							Panel B: RMSE						
1.02	0.5	0.65	100	**0.174**	0.182	0.201	0.203	0.214	0.149	0.147	**0.090**	0.112	0.211
1.02	0.5	0.65	200	**0.159**	0.161	0.162	0.175	0.198	0.124	0.127	**0.052**	0.086	0.189
1.02	0.5	0.65	400	0.154	0.155	**0.091**	0.132	0.154	0.111	0.113	**0.033**	0.055	0.134
1.02	0.4	0.6	100	0.203	0.197	**0.157**	0.172	0.247	0.146	0.154	**0.086**	0.126	0.205
1.02	0.4	0.6	200	0.201	0.199	**0.123**	0.156	0.216	0.119	0.127	**0.047**	0.091	0.166
1.02	0.4	0.6	400	0.200	0.200	**0.077**	0.126	0.158	0.107	0.111	**0.023**	0.058	0.103
1.05	0.5	0.65	100	0.155	0.156	0.150	0.175	**0.148**	0.125	0.142	**0.049**	0.084	0.132
1.05	0.5	0.65	200	0.151	0.151	**0.063**	0.113	0.104	0.107	0.121	**0.017**	0.042	0.070
1.05	0.5	0.65	400	0.150	0.150	**0.031**	0.056	0.067	0.102	0.108	**0.008**	0.017	0.040
1.05	0.4	0.6	100	0.201	0.199	**0.119**	0.153	0.163	0.120	0.146	**0.036**	0.094	0.111
1.05	0.4	0.6	200	0.200	0.200	**0.062**	0.117	0.119	0.107	0.120	**0.014**	0.052	0.058
1.05	0.4	0.6	400	0.200	0.200	**0.036**	0.059	0.069	0.102	0.108	**0.005**	0.016	0.023

Note: See notes to Table B.2.

Table B.9: Bias and RMSE of AR(1) estimates (single bubble; $u_t = e_t + 0.5e_{t-1}$).

δ	λ_1^0	λ_2^0	T	$\widehat{\delta}^{JS}$	$\widehat{\delta}^{PQ}$	$\widetilde{\delta}^{JS}$	$\widetilde{\delta}^{PQ}$	$\widehat{\delta}^{PSY}$
				Panel A: Bias				
1.02	0.5	0.65	100	−0.677	−0.627	**−0.132**	−0.236	−0.211
1.02	0.5	0.65	200	−0.735	−0.717	**−0.058**	−0.183	−0.172
1.02	0.5	0.65	400	−0.778	−0.771	**−0.022**	−0.097	−0.102
1.02	0.4	0.6	100	−0.762	−0.729	**−0.127**	−0.294	−0.205
1.02	0.4	0.6	200	−0.809	−0.793	**−0.055**	−0.239	−0.161
1.02	0.4	0.6	400	−0.856	−0.852	**−0.017**	−0.159	−0.072
1.05	0.5	0.65	100	−0.934	−0.906	**−0.071**	−0.215	−0.135
1.05	0.5	0.65	200	−0.974	−0.969	**−0.021**	−0.085	−0.050
1.05	0.5	0.65	400	−1.012	−1.011	**−0.004**	−0.009	−0.032
1.05	0.4	0.6	100	−0.983	−0.969	**−0.065**	−0.280	−0.112
1.05	0.4	0.6	200	−1.005	−1.003	**−0.017**	−0.150	−0.040
1.05	0.4	0.6	400	−1.036	−1.036	**−0.002**	−0.018	−0.038
				Panel B: RMSE				
1.02	0.5	0.65	100	0.782	0.736	**0.206**	0.397	0.267
1.02	0.5	0.65	200	0.792	0.779	**0.093**	0.360	0.232
1.02	0.5	0.65	400	0.818	0.814	**0.041**	0.262	0.190
1.02	0.4	0.6	100	0.849	0.812	**0.198**	0.468	0.263
1.02	0.4	0.6	200	0.852	0.840	**0.089**	0.433	0.227
1.02	0.4	0.6	400	0.884	0.882	**0.034**	0.367	0.157
1.05	0.5	0.65	100	0.966	0.946	**0.135**	0.426	0.207
1.05	0.5	0.65	200	0.985	0.981	**0.046**	0.267	0.119
1.05	0.5	0.65	400	1.018	1.017	**0.014**	0.079	0.086
1.05	0.4	0.6	100	1.001	0.989	**0.125**	0.503	0.184
1.05	0.4	0.6	200	1.012	1.010	**0.037**	0.378	0.105
1.05	0.4	0.6	400	1.038	1.038	**0.008**	0.132	0.080

Note: See notes to Table B.3.

Table B.10: Probabilities of break date selection (two bubbles; $u_t = e_t + 0.5e_{t-1}$).

δ_1	T	T_i^0	$\widehat{p}_i^C(JS)$	$\widehat{p}_i^L(JS)$	$\widehat{p}_i^C(PQ)$	$\widehat{p}_i^L(PQ)$	$\widetilde{p}_i^C(JS)$	$\widetilde{p}_i^L(JS)$	$\widetilde{p}_i^C(PQ)$	$\widetilde{p}_i^L(PQ)$	$\widehat{p}_i^C(PSY)$	$\widehat{p}_i^L(PSY)$	$\widehat{p}_i^C(HLW)$	$\widehat{p}_i^L(HLW)$
Panel A: First break date [i=1]														
1.05	100	20	0.00	0.97	0.01	0.92	**0.06**	0.38	0.04	0.50	0.02	0.94	**0.06**	0.38
1.05	200	40	0.00	0.99	0.00	0.99	**0.06**	0.51	0.03	0.52	0.01	0.97	**0.06**	0.50
1.05	400	80	0.00	1.00	0.00	1.00	**0.08**	0.57	0.05	0.48	0.01	0.98	**0.08**	0.57
Panel B: Second break date [i=2]														
1.05	100	40	0.00	0.98	0.12	0.88	**0.97**	0.02	0.68	0.27	0.64	0.11	0.90	0.04
1.05	200	80	0.00	0.99	0.02	0.98	**0.99**	0.00	0.78	0.19	0.90	0.05	0.97	0.02
1.05	400	160	0.00	1.00	0.00	1.00	**1.00**	0.00	0.90	0.10	0.78	0.22	0.99	0.00
δ_2	T	T_i^0	$\widehat{p}_i^C(JS)$	$\widehat{p}_i^L(JS)$	$\widehat{p}_i^C(PQ)$	$\widehat{p}_i^L(PQ)$	$\widetilde{p}_i^C(JS)$	$\widetilde{p}_i^L(JS)$	$\widetilde{p}_i^C(PQ)$	$\widetilde{p}_i^L(PQ)$	$\widehat{p}_i^C(PSY)$	$\widehat{p}_i^L(PSY)$	$\widehat{p}_i^C(HLW)$	$\widehat{p}_i^L(HLW)$
Panel C: Third break date [i=3]														
1.05	100	60	0.00	0.98	0.02	0.89	**0.11**	0.41	0.09	0.51	0.01	0.91	**0.11**	0.41
1.05	200	120	0.00	0.99	0.00	0.98	**0.11**	0.48	0.08	0.50	0.01	0.94	**0.11**	0.49
1.05	400	240	0.00	1.00	0.00	1.00	**0.14**	0.51	0.09	0.45	0.00	0.97	**0.14**	0.50
Panel D: Fourth break date [i=4]														
1.05	100	80	0.00	0.98	0.18	0.81	**0.98**	0.01	0.80	0.20	0.84	0.01	0.92	0.03
1.05	200	160	0.00	0.99	0.03	0.96	**1.00**	0.00	0.89	0.11	0.92	0.04	0.97	0.01
1.05	400	320	0.00	1.00	0.00	1.00	**1.00**	0.00	0.96	0.04	0.77	0.22	0.99	0.00

Note: See notes to Table B.1.

Table B.11: Bias and RMSE of break fraction estimates (two bubbles; $u_t = e_t + 0.5e_{t-1}$).

δ_1	λ_1^0	λ_2^0	T	$\widehat{\lambda}_1^{JS}$	$\widehat{\lambda}_2^{JS}$	$\widetilde{\lambda}_1^{PQ}$	$\widetilde{\lambda}_2^{PQ}$	$\widetilde{\lambda}_1^{JS}$	$\widetilde{\lambda}_2^{JS}$	$\widetilde{\lambda}_1^{PQ}$	$\widetilde{\lambda}_2^{PQ}$	$\widehat{\lambda}_1^{PSY}$	$\widehat{\lambda}_2^{PSY}$	$\widehat{\lambda}_1^{HLW}$	$\widehat{\lambda}_2^{HLW}$
Panel A: Bias of first bubble estimators															
1.05	0.2	0.4	100	0.197	0.114	0.174	0.120	−0.010	0.001^c	0.039	0.032	0.119	0.012	-0.006^s	-0.001^c
1.05	0.2	0.4	200	0.200	0.105	0.195	0.115	0.005^s	0.000^c	0.031	0.019	0.099	0.008	0.008	0.003
1.05	0.2	0.4	400	0.201	0.102	0.200	0.107	0.014^s	0.000^c	0.016	0.009	0.060	0.002	0.015	0.001
Panel B: RMSE of first bubble estimators															
1.05	0.2	0.4	100	0.202	0.126	0.188	0.137	0.063^s	0.019^c	0.115	0.081	0.162	0.099	0.075	0.051
1.05	0.2	0.4	200	0.201	0.109	0.198	0.121	0.055^s	0.008^c	0.100	0.057	0.129	0.063	0.069	0.046
1.05	0.2	0.4	400	0.202	0.105	0.200	0.108	0.040^s	0.004^c	0.074	0.036	0.078	0.025	0.046	0.023
δ_2	T	λ_3^0	λ_4^0	$\widehat{\lambda}_3^{JS}$	$\widehat{\lambda}_4^{JS}$	$\widetilde{\lambda}_3^{PQ}$	$\widetilde{\lambda}_4^{PQ}$	$\widetilde{\lambda}_3^{JS}$	$\widetilde{\lambda}_4^{JS}$	$\widetilde{\lambda}_3^{PQ}$	$\widetilde{\lambda}_4^{PQ}$	$\widehat{\lambda}_3^{PSY}$	$\widehat{\lambda}_4^{PSY}$	$\widehat{\lambda}_3^{HLW}$	$\widehat{\lambda}_4^{HLW}$
Panel C: Bias of second bubble estimators															
1.05	0.6	0.8	100	0.190	0.091	0.159	0.080	-0.007^s	0.000^c	0.026	0.019	0.070	−0.026	−0.010	−0.009
1.05	0.6	0.8	200	0.197	0.097	0.192	0.096	0.005^s	0.000^c	0.016	0.011	0.068	−0.007	0.007	−0.001
1.05	0.6	0.8	400	0.199	0.099	0.199	0.100	0.010	0.000^c	0.006^s	0.004	0.046	0.000^c	0.010	0.000^c
Panel D: RMSE of second bubble estimators															
1.05	0.6	0.8	100	0.200	0.108	0.182	0.091	0.059^s	0.013^c	0.107	0.045	0.118	0.097	0.075	0.051
1.05	0.6	0.8	200	0.200	0.102	0.197	0.099	0.049^s	0.006^c	0.085	0.033	0.102	0.063	0.069	0.046
1.05	0.6	0.8	400	0.200	0.101	0.200	0.100	0.036^s	0.002^c	0.056	0.020	0.064	0.021	0.046	0.023

Note: See notes to Table B.5.

Table B.12: Bias and RMSE of AR(1) estimates (two bubbles; $u_t = e_t + 0.5e_{t-1}$).

δ	T	$\widehat{\delta}_1^{JS}$	$\widehat{\delta}_1^{PQ}$	$\widetilde{\delta}_1^{JS}$	$\widetilde{\delta}_1^{PQ}$	$\widehat{\delta}_1^{PSY}$	$\widehat{\delta}_1^{HLW}$	$\widehat{\delta}_2^{JS}$	$\widehat{\delta}_2^{PQ}$	$\widetilde{\delta}_2^{JS}$	$\widetilde{\delta}_2^{PQ}$	$\widehat{\delta}_2^{PSY}$	$\widehat{\delta}_2^{HLW}$
Panel A: Bias													
1.05	100	−0.960	−0.870	−**0.070**	−0.300	−0.300	−**0.070**	−1.010	−0.840	−**0.060**	−0.230	−0.090	−0.110
1.05	200	−0.990	−0.980	−**0.020**	−0.200	−0.200	−**0.020**	−1.010	−0.980	−**0.020**	−0.120	−0.030	−0.030
1.05	400	−1.020	−1.030	**0.000**	−0.100	−0.100	**0.000**	−1.040	−1.030	**0.000**	−0.040	−0.040	**0.000**
Panel B: RMSE													
1.05	100	1.000	0.940	**0.140**	0.520	0.520	**0.140**	1.020	0.920	**0.110**	0.450	0.180	0.270
1.05	200	1.000	0.990	**0.050**	0.440	0.440	**0.050**	1.010	1.000	**0.040**	0.330	0.100	0.120
1.05	400	1.030	1.030	**0.010**	0.310	0.310	**0.010**	1.040	1.040	**0.010**	0.210	0.080	0.050

Note: See notes to Table B.3.

Table B.13: Mean of break fraction estimates for ($\hat{m} = 4 | m_0 = 2$) replications.

Panel A: Mean of break fraction estimates for first estimated bubble															
δ	λ_1^0	λ_2^0	T	$\hat{\lambda}_1^{JS}$	$\hat{\lambda}_2^{JS}$	$\hat{\lambda}_1^{PQ}$	$\hat{\lambda}_2^{PQ}$	$\tilde{\lambda}_1^{JS}$	$\tilde{\lambda}_2^{JS}$	$\tilde{\lambda}_1^{PQ}$	$\tilde{\lambda}_2^{PQ}$	$\hat{\lambda}_1^{PSY}$	$\hat{\lambda}_2^{PSY}$	$\hat{\lambda}_1^{HLW}$	$\hat{\lambda}_2^{HLW}$
1.02	0.5	0.65	100	0.263	0.470	0.232	0.460	0.235	0.448	0.213	0.438	0.406	0.482	0.256	0.439
1.02	0.5	0.65	200	0.260	0.448	0.259	0.472	0.227	0.410	0.217	0.427	0.397	0.450	0.256	0.428
1.02	0.5	0.65	400	0.266	0.438	0.292	0.486	0.231	0.391	0.230	0.427	0.404	0.442	0.267	0.428
1.02	0.4	0.6	100	0.253	0.453	0.228	0.462	0.239	0.471	0.212	0.442	0.394	0.470	0.249	0.430
1.02	0.4	0.6	200	0.255	0.431	0.249	0.463	0.240	0.442	0.217	0.427	0.389	0.445	0.248	0.416
1.02	0.4	0.6	400	0.280	0.431	0.274	0.463	0.258	0.426	0.240	0.430	0.375	0.422	0.246	0.404
1.05	0.5	0.65	100	0.262	0.481	0.249	0.496	0.234	0.426	0.222	0.457	0.381	0.457	0.255	0.432
1.05	0.5	0.65	200	0.286	0.465	0.292	0.508	0.236	0.397	0.232	0.443	0.374	0.432	0.269	0.430
1.05	0.5	0.65	400	0.355	0.492	0.334	0.525	0.235	0.386	0.254	0.448	0.365	0.416	0.282	0.429
1.05	0.4	0.6	100	0.262	0.460	0.237	0.482	0.239	0.456	0.215	0.456	0.368	0.446	0.244	0.418
1.05	0.4	0.6	200	0.310	0.456	0.270	0.479	0.254	0.421	0.235	0.437	0.354	0.421	0.253	0.411
1.05	0.4	0.6	400	0.364	0.482	0.301	0.487	0.266	0.424	0.270	0.447	0.325	0.398	0.258	0.410
Panel B: Mean of break fraction estimates for second estimated bubble															
δ	λ_1^0	λ_2^0	T	$\hat{\lambda}_3^{JS}$	$\hat{\lambda}_4^{JS}$	$\hat{\lambda}_3^{PQ}$	$\hat{\lambda}_4^{PQ}$	$\tilde{\lambda}_3^{JS}$	$\tilde{\lambda}_4^{JS}$	$\tilde{\lambda}_3^{PQ}$	$\tilde{\lambda}_4^{PQ}$	$\hat{\lambda}_3^{PSY}$	$\hat{\lambda}_4^{PSY}$	$\hat{\lambda}_3^{HLW}$	$\hat{\lambda}_4^{HLW}$
1.02	0.5	0.65	100	0.645	0.790	0.650	0.808	0.603	0.761	0.622	0.793	0.629	0.710	0.622	0.782
1.02	0.5	0.65	200	0.647	0.768	0.664	0.797	0.564	0.712	0.614	0.765	0.623	0.691	0.598	0.741
1.02	0.5	0.65	400	0.649	0.756	0.671	0.787	0.549	0.690	0.610	0.744	0.630	0.696	0.597	0.735
1.02	0.4	0.6	100	0.610	0.752	0.637	0.797	0.619	0.771	0.616	0.790	0.612	0.694	0.599	0.756
1.02	0.4	0.6	200	0.604	0.720	0.641	0.776	0.583	0.730	0.606	0.757	0.589	0.663	0.573	0.717
1.02	0.4	0.6	400	0.602	0.708	0.642	0.762	0.566	0.713	0.608	0.740	0.583	0.668	0.557	0.704
1.05	0.5	0.65	100	0.650	0.778	0.668	0.811	0.572	0.718	0.626	0.783	0.616	0.705	0.591	0.734
1.05	0.5	0.65	200	0.650	0.761	0.676	0.800	0.549	0.692	0.613	0.756	0.626	0.712	0.591	0.730
1.05	0.5	0.65	400	0.650	0.754	0.684	0.797	0.547	0.685	0.612	0.742	0.632	0.722	0.601	0.739
1.05	0.4	0.6	100	0.605	0.733	0.648	0.796	0.592	0.741	0.621	0.785	0.582	0.679	0.567	0.714
1.05	0.4	0.6	200	0.603	0.713	0.649	0.778	0.558	0.712	0.607	0.751	0.589	0.693	0.560	0.713
1.05	0.4	0.6	400	0.601	0.704	0.658	0.776	0.561	0.717	0.614	0.748	0.593	0.703	0.561	0.723

Note: See notes to Table B.2.

Table B.14: Standard deviation of break fraction estimates for $(\hat{m} = 4|m_0 = 2)$ replications.

Panel A: Standard deviation of break fraction estimates for first estimated bubble															
δ	λ_1^0	λ_2^0	T	$\hat{\lambda}_1^{JS}$	$\hat{\lambda}_2^{JS}$	$\hat{\lambda}_1^{PQ}$	$\hat{\lambda}_2^{PQ}$	$\tilde{\lambda}_1^{JS}$	$\tilde{\lambda}_2^{JS}$	$\tilde{\lambda}_1^{PQ}$	$\tilde{\lambda}_2^{PQ}$	$\hat{\lambda}_1^{PSY}$	$\hat{\lambda}_2^{PSY}$	$\hat{\lambda}_1^{HLW}$	$\hat{\lambda}_2^{HLW}$
1.02	0.5	0.65	100	0.059	0.090	0.056	0.112	0.060	0.133	0.046	0.106	0.146	0.167	0.074	0.120
1.02	0.5	0.65	200	0.058	0.086	0.070	0.111	0.067	0.127	0.050	0.101	0.147	0.165	0.079	0.120
1.02	0.5	0.65	400	0.059	0.083	0.078	0.107	0.075	0.117	0.058	0.100	0.148	0.167	0.089	0.128
1.02	0.4	0.6	100	0.063	0.081	0.055	0.105	0.059	0.124	0.048	0.104	0.148	0.170	0.071	0.115
1.02	0.4	0.6	200	0.059	0.075	0.063	0.107	0.067	0.133	0.055	0.106	0.128	0.145	0.067	0.106
1.02	0.4	0.6	400	0.058	0.072	0.069	0.108	0.074	0.134	0.075	0.116	0.117	0.141	0.069	0.113
1.05	0.5	0.65	100	0.057	0.079	0.063	0.105	0.069	0.131	0.052	0.109	0.112	0.132	0.071	0.108
1.05	0.5	0.65	200	0.065	0.082	0.078	0.101	0.079	0.119	0.062	0.105	0.121	0.146	0.088	0.125
1.05	0.5	0.65	400	0.070	0.077	0.085	0.101	0.080	0.115	0.075	0.108	0.117	0.158	0.100	0.141
1.05	0.4	0.6	100	0.065	0.070	0.058	0.103	0.063	0.132	0.049	0.106	0.096	0.120	0.060	0.101
1.05	0.4	0.6	200	0.060	0.067	0.064	0.105	0.072	0.136	0.068	0.114	0.099	0.133	0.070	0.119
1.05	0.4	0.6	400	0.042	0.046	0.071	0.111	0.074	0.139	0.091	0.129	0.087	0.144	0.072	0.131
Panel B: Standard deviation of break fraction estimates for second estimated bubble															
δ	λ_1^0	λ_2^0	T	$\hat{\lambda}_3^{JS}$	$\hat{\lambda}_4^{JS}$	$\hat{\lambda}_3^{PQ}$	$\hat{\lambda}_4^{PQ}$	$\tilde{\lambda}_3^{JS}$	$\tilde{\lambda}_4^{JS}$	$\tilde{\lambda}_3^{PQ}$	$\tilde{\lambda}_4^{PQ}$	$\hat{\lambda}_3^{PSY}$	$\hat{\lambda}_4^{PSY}$	$\hat{\lambda}_3^{HLW}$	$\hat{\lambda}_4^{HLW}$
1.02	0.5	0.65	100	0.072	0.097	0.097	0.107	0.157	0.162	0.111	0.132	0.228	0.247	0.162	0.207
1.02	0.5	0.65	200	0.051	0.068	0.080	0.094	0.148	0.152	0.108	0.128	0.209	0.213	0.163	0.195
1.02	0.5	0.65	400	0.028	0.039	0.072	0.087	0.129	0.134	0.108	0.114	0.204	0.190	0.169	0.198
1.02	0.4	0.6	100	0.070	0.111	0.094	0.116	0.160	0.180	0.108	0.136	0.248	0.268	0.171	0.225
1.02	0.4	0.6	200	0.046	0.071	0.086	0.115	0.176	0.194	0.111	0.138	0.221	0.219	0.173	0.219
1.02	0.4	0.6	400	0.034	0.046	0.084	0.114	0.177	0.194	0.123	0.142	0.229	0.203	0.183	0.220
1.05	0.5	0.65	100	0.031	0.071	0.075	0.092	0.151	0.157	0.111	0.133	0.191	0.196	0.150	0.187
1.05	0.5	0.65	200	0.017	0.038	0.071	0.089	0.132	0.136	0.111	0.126	0.206	0.192	0.167	0.196
1.05	0.5	0.65	400	0.008	0.017	0.075	0.092	0.122	0.128	0.116	0.121	0.228	0.191	0.183	0.215
1.05	0.4	0.6	100	0.047	0.083	0.088	0.111	0.178	0.196	0.111	0.137	0.222	0.226	0.169	0.216
1.05	0.4	0.6	200	0.033	0.049	0.086	0.114	0.182	0.194	0.125	0.143	0.250	0.227	0.197	0.236
1.05	0.4	0.6	400	0.017	0.023	0.090	0.120	0.186	0.195	0.141	0.152	0.272	0.221	0.213	0.246

Note: See notes to Table B.2.

Table B.15: Mean and standard deviation of break fraction estimates for $(\hat{m} = 2|m_0 = 4)$ replications.

Panel A: Mean of break fraction estimates																	
δ	λ_1^0	λ_2^0	λ_3^0	λ_4^0	T	$\hat{\lambda}_1^{JS}$	$\hat{\lambda}_2^{JS}$	$\hat{\lambda}_1^{PQ}$	$\hat{\lambda}_2^{PQ}$	$\tilde{\lambda}_1^{JS}$	$\tilde{\lambda}_2^{JS}$	$\tilde{\lambda}_1^{PQ}$	$\tilde{\lambda}_2^{PQ}$	$\hat{\lambda}_1^{PSY}$	$\hat{\lambda}_2^{PSY}$	$\hat{\lambda}_1^{HLW}$	$\hat{\lambda}_2^{HLW}$
1.05	0.2	0.4	0.6	0.8	100	0.639	0.755	0.534	0.752	0.419	0.653	0.406	0.769	0.526	0.626	0.419	0.653
1.05	0.2	0.4	0.6	0.8	200	0.632	0.737	0.597	0.743	0.437	0.635	0.406	0.742	0.520	0.632	0.437	0.635
1.05	0.2	0.4	0.6	0.8	400	0.630	0.732	0.627	0.736	0.452	0.631	0.407	0.679	0.489	0.631	0.452	0.631
Panel B: Standard deviation of break fraction estimates																	
1.05	0.2	0.4	0.6	0.8	100	0.235	0.259	0.220	0.207	0.157	0.231	0.078	0.126	0.203	0.242	0.157	0.231
1.05	0.2	0.4	0.6	0.8	200	0.237	0.271	0.239	0.248	0.164	0.236	0.067	0.155	0.196	0.238	0.164	0.236
1.05	0.2	0.4	0.6	0.8	400	0.237	0.275	0.238	0.269	0.168	0.237	0.070	0.194	0.184	0.238	0.168	0.237

Note: See note to Table B.2.