Estimating Failure Time Distribution and Its Parameters Based on Intermediate Data from a Wiener Degradation Model

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Abstract: Instead of measuring a Wiener degradation or performance process at predetermined time points to track degradation or performance of a product for estimating its lifetime, we propose to obtain the first-passage times of the process over certain nonfailure thresholds. Based on only these intermediate data, we obtain the uniformly minimum variance unbiased estimator and uniformly most accurate confidence interval for the mean lifetime. For estimating the lifetime distribution function, we propose a modified maximum likelihood estimator and a new estimator and prove that, by increasing the sample size of the intermediate data, these estimators and the above-mentioned estimator of the mean lifetime can achieve the same levels of accuracy as the estimators assuming one has failure times. Thus, our method of using only intermediate data is useful for highly reliable products when their failure times are difficult to obtain. Furthermore, we show that the proposed new estimator of the lifetime distribution function is more accurate than the standard and modified maximum likelihood estimators. We also obtain approximate confidence intervals for the lifetime distribution function and its percentiles. Finally, we use light-emitting diodes as an example to illustrate our method and demonstrate how to validate the Wiener assumption during the testing. © 2008 Wiley Periodicals, Inc. Naval Research Logistics 55: 265–276, 2008

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1. INTRODUCTION

To face intensive global competition and meet higher customer expectations, today's manufacturers are under pressure to design products with high reliability. At the same time, however, these manufacturers cannot afford to conduct long tests to obtain failure time (or lifetime) data for estimating their products' lifetime distributions. This raises two important problems for managers, reliability engineers, and statisticians: (1) how to design an efficient sampling scheme for collecting reliability data, and (2) how to develop statistical methods to analyze these data for estimating the lifetime in a timely manner.

A standard approach to quickly obtaining lifetime data is to use an accelerated life test (ALT), in which higher stress conditions are applied to products either from the beginning of, or gradually during, the testing. The goal is to accelerate failures of the test units so that failure times can be obtained sooner. Because this approach and the corresponding analysis methods are rather well-developed, the reader is referred to, for example, see [17], for detail.

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A more recent approach is to use a degradation test (DT), where one will assume a degradation model and a degradation or performance process for the product. First, degradation may be defined as a phenomenon that takes in systems and structures subjected to stress. To many engineers, degradation is the accumulation of damage leading eventually to a weakness that can cause failure. One degradation model assumes that it is possible to define and measure degradation or damage (level) of a product as a function of time. Such measures can provide more information than failure-time data for the purpose of assessing and improving product reliability [16]. On the other hand, when degradation can be modeled by a process but is latent, Whitmore et al. [27] and Lee et al. [14] propose a model with a bivariate Wiener process, say $\{(H(t), W(t))|t \geq 0\}$, where $H(t)$ represents the latent degradation level, whereas $W(t)$ is the value of a correlated, observable (marker) process at time t. Then, data on $W(t)$ collected at some predetermined time points during a DT are used to estimate the true lifetime, which is the first hitting time of $H(t)$ to a threshold. Recently, Singpurwalla [21] considers a case where $\{W(t)|t \geq 0\}$ is assumed to be an observable Wiener process (because cracks do heal and CD4 blood cell counts do fluctuate) with a link to the unobservable $\{H(t)|t \geq 0\}$ described by

$$H(t) = \sup\{W(s)|0 \leq s \leq t\}.$$

$\{H(t)\}$ is called a Wiener Maximum Process. An item experiencing the process $H(t)$ fails when $H(t)$ first crosses a failure threshold a, which can be unknown or random. Singpurwalla then shows that the first hitting time of $H(t)$ to a coincides with the first hitting time of $W(t)$ to a. By symmetry, if $W(t) = \sup\{H(s)|0 \leq s \leq t\}$ with unobservable Wiener $H(t)$, then Singpurwalla's claim above is also true. This model can be used to model, for example, metal fatigue when crack length, $W(t)$, is assumed to be nondecreasing in t.

For some products, however, "damage" is difficult to define explicitly, although the exact phenomenon that causes or promotes degradation is known. Metal fatigue as described by Ebrahimi [7] and degradation of a light-emitting diode (LED) by Zhao and Elsayed [29] are examples. Nevertheless, it is claimed that damage manifests itself via observable surrogates such as cracks, decrease in light intensity, corrosion, and measured wear. Then, with damage being difficult to define and model, a standard approach in the degradation data analysis literature is to work only with this surrogate or a surrogate process. Different names have been given to this surrogate process; for example, performance [17, Chapter 11], degradation signal [8], performance degradation process [29], or simply degradation process [16, Chapter 13]. An item is defined to "fail" when its surrogate measure first reaches a specific threshold. This first hitting time is then used as an approximation of the true failure time. It is also possible to study degradation based on the hard rate approach [4].

The method (along with the results in Appendix D for random thresholds) that we will propose for obtaining timely data and the corresponding estimators applies to the first model with a measurable degradation process and the third with a known performance process. Our method also applies to Singpurwalla's bivariate model, but not the bivariate setting of Whitmore et al. [27] and Lee et al. [14] unless we also define an unobservable degradation process with a specific link to the observable one. The first and the third models are commonly seen in engineering applications, whereas the bivariate model has applications in the biostatistical field [4]. In any case, we will call our process a degradation process throughout this article, although it can be considered a performance process as well.

After a degradation model is defined, the degradation process is typically described by (i) a mixed-effect nonlinear regression or (ii) a stochastic process. In the mixed-effect regression of Lu and Meeker [15] and Meeker and Escobar [16], the dependent variable is the degradation measure, and the independent variables (effects) are time and other covariates. The random effect terms reflect the individual product characteristics, whereas the fixed effects are common for all units. The random error term represents measurement errors. As to the analysis, the approximate maximum likelihood principle is generally followed when estimating the failure time distribution function, whereas simulation-based approaches or asymptotic results are used to obtain approximate confidence intervals for its percentiles.

In this article, we will use (ii) by assuming that the degradation process is a (time-transformed) Wiener process and that failure occurs when this process first reaches a constant failure threshold. In addition to the stochastic processes mentioned earlier, Tang and Chang [23] model their nondestructive, accelerated degradation of power supply units as a class of stochastic processes. The simple Wiener process that we assume [in (1) of Section 2] has been used by Hasofer [9] in his study of emptiness of a dam, and by Banerjee and Bhattacharyya [1] in their study of a purchase incident model. Lancaster [11] uses it when studying duration of strikes. Sheppard [20] uses it in his study of an injected labeled substance, called tracer, in a biological system. More recently, Whitmore and Schenkelberg [28] use a time-transformed Wiener process to model resistance of self-regulating heating cables. Doksum and Normand [6] assume a Wiener process for the level of a biomarker process, such as calibrated log CD4 blood cell counts, in their HIV study. Liao and Elsayed [13] and Tseng et al. [24] also use a simple Wiener process for the (linearized) light intensity of LED lamps of contact image scanners (CISs). Doksum and Hoyland [5] model an accelerated degradation path with a time-transformed Wiener process. In contrast to (i) where the failure time distribution function is assumed, the distribution under (ii) is determined by the type of process used.

The article is organized as follows. Section 2 briefly describes the Wiener process and its threshold-crossing time distribution; namely, the inverse Gaussian (IG) distribution. Then, in Section 3, we propose to obtain the first-passage times of the degradation process over certain predetermined, constant nonfailure thresholds during the early stage of a DT. The data will be called intermediate data, and they also follow IG distributions. Through the similarity of these IG distributions, we obtain in Section 4.1 the maximum likelihood estimator (MLE), uniformly minimum variance unbiased estimator (UMVUE), and the uniformly most accurate (UMA) confidence interval, all based on intermediate data, for the product's mean lifetime. We then prove that, by increasing the sample size of the intermediate data, these estimators can achieve the same levels of accuracy as the estimators, assuming one has actual failure times. Section 4.2 gives the UMVUE of the failure time distribution function in case one also has failure time data, which is often the case when a DT is accelerated. For the failure time distribution function, we propose a modified MLE (MMLE, Section 4.3.1) and another new estimator (Section 4.3.2), all based only on intermediate data. We then prove that, by increasing the sample size of the intermediate data, this MMLE can be as efficient as the estimator using actual failure times (if any). Furthermore, we show in Section 4.3.2 that our proposed new estimator is asymptotically more accurate than the traditional MLE and the MMLE of Section 4.3.1. Approximate or asymptotic confidence intervals for the lifetime distribution function and its percentiles are given in Section 4.4. In Section 5, we illustrate our results using an example on LED lamps for CISs from Tseng et al. [24]. We also demonstrate how we can use intermediate data to validate the key Wiener assumption for the degradation process during the testing. Section 6 presents our concluding remarks.

2. DEGRADATION PROCESS AND FAILURE TIME DISTRIBUTION

Assume that the (transformed) degradation process, $W(t)$, of a test unit is Wiener:

$$W(t) = \eta t + \sigma B(t), \quad t \geq 0, \tag{1}$$

where η is the drift rate, $\sigma > 0$ is the diffusion coefficient, and $B(t)$ is the standard Brownian motion. The values of the unknown η and σ are assumed to be common for all units, which is a strong assumption. On the other hand, uniformity of production units may be assured with robust process/product designs and effective manufacturing process control, especially for high-volume products made by automatic equipment. At each t, $W(t)$ is distributed as normal $(\eta t, \sigma^2 t)$, and because $\mathrm{Cov}(B(t), B(s)) = \min(s,t)\sigma^2 > 0$, $W(t)$ is a positively correlated process. The process (1) is the solution of the stochastic linear growth (cumulative decay) model [with $W(0) = 0$]:

$$dW(t) = \eta dt + \sigma dB(t),$$

which reduces to a deterministic linear growth model when $\sigma = 0$. In addition to the numerous applications mentioned in Section 1, $W(t)$ is also used as a continuous-time approximation of the discrete-time cumulative sum (CUSUM) process with drift and has applications in statistical quality control [22] and finance. A geometric Brownian motion $S(t)$, which is a solution of the stochastic exponential growth model $dS(t) = \gamma S(t)dt + \sigma S(t)dB(t)$, has been used in degradation data analysis; see, for example, Ebrahimi [7]. By taking the logarithm of $S(t)$, one obtains (1) except for a constant.

For a test unit whose degradation process $W(t)$ satisfies (1) with $\eta > 0$ and $W(0) = w_0$, its failure time is defined as the first-passage time of $W(t)$ over a constant failure threshold, say $a(> w_0)$. Because a Wiener process is a Markov process with independent stationary increments, the failure time distribution function depends on a and w_0 through $a - w_0 (> 0)$. Hence, we may assume $w_0 = 0$, and the failure time of the test unit is then

$$T_a = \inf\{t \geq 0 | W(0) = 0, \quad W(t) \geq a\}. \tag{2}$$

It is well-known that lifetime T_a follows an IG distribution, denoted by $IG(\mu, \lambda)$, with probability density function (pdf) and cumulative distribution function (cdf)

$$f(t_a) = \sqrt{\frac{\lambda}{2\pi}}\, t_a^{-3/2} \exp\left\{-\frac{\lambda(t_a - \mu)^2}{2\mu^2 t_a}\right\}, \quad t_a > 0, \tag{3}$$

$$F(t_a) = \Phi\left[\sqrt{\frac{\lambda}{t_a}}\left(\frac{t_a}{\mu} - 1\right)\right] + e^{2\lambda/\mu} \times \Phi\left[-\sqrt{\frac{\lambda}{t_a}}\left(\frac{t_a}{\mu} + 1\right)\right], \quad t_a > 0, \tag{4}$$

respectively, where the mean μ and scale parameter λ satisfy

$$\mu = \frac{a}{\eta}, \quad \text{and} \quad \lambda = \frac{a^2}{\sigma^2}. \tag{5}$$

The IG distribution in (4) constitutes an exponential family. If only failure time data are available, MLEs and UMVUEs of μ and λ and some of their functions have been obtained [10, Table 1].

3. THE PROPOSED SAMPLING SCHEME AND INTERMEDIATE DATA

To estimate the lifetime distribution function and its parameters, we propose that, for each test unit, we measure the first-passage times, denoted $T_1, \ldots, T_m$, of its degradation process $W(t)$ over certain predetermined nonfailure thresholds $a_1, \ldots, a_m$, respectively, where $0 < a_1 < \cdots < a_m < a$; that is,

$$T_j = \inf\{t \geq 0 | W(0) = 0, \quad W(t) \geq a_j\}, \quad j = 1, \ldots, m. \tag{6}$$

We have $0 < T_1 < \cdots < T_m < T_a$ with probability one. We will obtain failure times as well if we assume $a_m = a$.

Similar to T_a, T_j follows $IG(a_j/\eta, a_j^2/\sigma^2)$. In our case, equally spaced a_j's seem reasonable because the expected path of $W(t)$ is linear in t. Because the T_j's are dependent, we consider the stationary and independent increments

$$D_j \equiv T_j - T_{j-1}, \quad \text{for } j = 1, \ldots, m, \quad \text{with } T_0 = 0. \quad (7)$$

Then, the conditional and unconditional distribution of D_j, given T_{j-1}, is $IG(\mu_j, \lambda_j)$ with $(a_0 = 0)$

$$\mu_j = \frac{a_j - a_{j-1}}{\eta} \quad \text{and} \quad \lambda_j = \frac{(a_j - a_{j-1})^2}{\sigma^2}. \quad (8)$$

Define

$$r_j = \frac{a}{a_j - a_{j-1}} \quad \text{and} \quad r = \frac{a}{a_m}, \quad (9)$$

then, $r^{-1} = \sum_{j=1}^{m} r_j^{-1}$ and from (5), (8), and (9), we have

$$\mu_j = r_j^{-1}\mu \quad \text{and} \quad \lambda_j = r_j^{-2}\lambda, \quad \text{for } j = 1, 2, \ldots, m. \quad (10)$$

Now, assume that n units are tested, and the first-passage times over the thresholds $a_1, \ldots, a_m$ for the ith unit are denoted by $T_{i1}, \ldots, T_{im}$, respectively. For a_j with $a_j < a$, $T_{j1}, \ldots, T_{jm}$ are called intermediate (nonfailure) data. We have the following three scenarios:

(a) When $m = 1$ and $a_m = a$, we have only failure time data, as in most LTs or ALTs.
(b) When $a_m < a$, we have only intermediate data. This is a likely case when dealing with highly reliable products in an unaccelerated DT.
(c) When $m \geq 2$ and $a_m = a$, we have both intermediate and failure time data, which is often the case in an ADT.

Because many results for case (a) are available [2, 19], this leaves only cases (b) and (c) for study. Obviously, with the same sample sizes, estimators under (c) are expected to perform better than the corresponding estimators under (b). On the other hand, if one can increase the sample size and does not wish to accelerate the DT (to avoid making assumptions about such relationships as Arrhenius and Eyring laws for extrapolating the accelerated results to obtain the results under normal operating conditions) for highly reliable products, our method of using only intermediate data in (b) can be very useful for obtaining efficient estimates of the lifetime distribution and its parameters. We will explain this in more detail in the next section.

4. THE ANALYSES

Given the data $T_{ij} = t_{ij}$, $i = 1, \ldots, n$, and $j = 1, \ldots, m$, and from the fact that $T_{ij} - T_{i,j-1}$ are independently distributed as $IG(\mu_j, \lambda_j)$, the likelihood function is [from (3)]

$$L = \left(\frac{\lambda}{2\pi}\right)^{nm/2} \left(\prod_{j=1}^{m} \frac{1}{r_j^n}\right) \left(\prod_{i=1}^{n}\prod_{j=1}^{m} (t_{ij} - t_{i,j-1})^{-3/2}\right) \times \exp\left[-\frac{\lambda}{2\mu^2} \sum_{i=1}^{n}\sum_{j=1}^{m} \frac{(t_{ij} - t_{i,j-1} - r_j^{-1}\mu)^2}{t_{ij} - t_{i,j-1}}\right], \quad (11)$$

from which a complete set of sufficient statistics for (μ, λ) is $(\sum_{i=1}^{n} T_{im}, \sum_{i=1}^{n}\sum_{j=1}^{m}(r_j^2(T_{ij} - T_{i,j-1}))^{-1})$.

4.1. Estimation of Mean Lifetime

To estimate the mean lifetime, we have (proof in Appendix A):

PROPOSITION 1: Let $\overline{T}_m \equiv \sum_{i=1}^{n} T_{im}/n$ and $V \equiv \sum_{i=1}^{n}\sum_{j=1}^{m}[r^2(r_j^2(T_{ij} - T_{i,j-1}))^{-1} - (m\overline{T}_m)^{-1}]$, then $\overline{T}_m$ and V are independent such that $\overline{T}_m$ is distributed as $IG(r^{-1}\mu, nr^{-2}\lambda)$ and $(r^{-2}\lambda)V$ as χ^2_{nm-1}. The MLE and UMVUE of the mean lifetime μ are both $\hat{\mu} = r\overline{T}_m$ with variance $(r/n)(\mu^3/\lambda)$. The MLE and UMVUE of $1/\lambda$ are $r^{-2}V(nm)$ and $r^{-2}V/(nm-1)$, respectively, with variances $2(nm-1)/(\lambda^2 n^2 m^2)$ and $2/[\lambda^2(nm-1)]$.

We have the following remarks. First, note that $r\overline{T}_m = r\sum_{i=1}^{n} T_{im}/n$ is the UMVUE of μ when only intermediate data are available (case (b) of Section 3), whereas $\overline{T}_a (= \sum_{i=1}^{n} T_{ia}/n)$ is the UMVUE, assuming we were to have the actual lifetimes (cases (a) and (c), for which $r\overline{T}_m = \overline{T}_a$). From Proposition 4.1

$$\mathrm{Var}(r\overline{T}_m) = r\frac{\mu^3}{n\lambda} = r\,\mathrm{Var}(\overline{T}_a). \quad (12)$$

Because standard deviations are used in mostly statistical inference (e.g., constructing confidence intervals or computing sampling errors), the relative efficiency of $r\overline{T}_m$ to $\overline{T}_a$ can reasonably be defined as

$$\frac{\sqrt{\mathrm{Var}(\overline{T}_a)}}{\sqrt{\mathrm{Var}(r\overline{T}_m)}} = \frac{1}{\sqrt{r}} = \sqrt{\frac{a_m}{a}}. \quad (13)$$

Note that initially (13) increases rapidly with $a_m (0 < a_m \leq a)$, but soon levels off. This implies that marginal gains in the efficiency of the unbiased estimate of μ, by prolonging the DT, decreases rapidly. Second, from Proposition 4.1, the standard deviation of the MLE/UMVUE of μ, based on only intermediate data ($a_m < a$) with size kn, is $\sqrt{r\mu^3/nk\lambda}$, while the standard deviation of the MLE/UMVUE, based on failure times ($a_m = a$) with size n, is $\sqrt{\mu^3/n\lambda}$. Therefore, the MLE/UMVUE based on only intermediate data is at least as efficient as the MLE/UMVUE based on actual lifetimes, provided $k \geq r (= a/a_m)$. For example, if we are able to double the current sample size, we can use $a_m = a/2$ (i.e., we hope to terminate the DT halfway to the expected lifetime) to obtain an efficient estimator of μ. In a typical ADT, one needs to assume models, such as Arrhenius' and Eyring's and then estimate their parameters to extrapolate the accelerated results to obtain the results under normal operating conditions. If it is feasible to lower a_j's and further increase the sample size, our estimator of the mean lifetime and also estimators in Section 4.3 for the lifetime distribution function can be efficient without accelerating a DT. The additional estimation step in an ADT may increase the variance of its estimators.

When λ is unknown, it can be shown that the $100(1-\alpha)\%$ UMA or UMA-unbiased confidence interval for μ is (cf. [2, (6.21)])

$$\left(r\overline{T}_m \left[1 + t_{1-\alpha/2,nm-1} \sqrt{\frac{\overline{T}_m V}{n(nm-1)}} \right]^{-1}, \; r\overline{T}_m \left[1 - t_{1-\alpha/2,nm-1} \sqrt{\frac{\overline{T}_m V}{n(nm-1)}} \right]^{-1} \right),$$
$$\text{if } 1 - t_{1-\alpha/2,nm-1}\sqrt{\overline{T}_m V/[n(nm-1)]} > 0, \quad (14)$$

and

$$\left(r\overline{T}_m \left[1 + t_{1-\alpha/2,nm-1} \sqrt{\frac{\overline{T}_m V}{n(nm-1)}} \right]^{-1}, \infty \right), \quad \text{otherwise.}$$

4.2. The UMVUE of *F(t)* with Both Intermediate and Failure Time Data

When a DT is accelerated, one is likely to have both intermediate and failure time data (case (c) in Section 3). Because $a_m = a$ and $m \geq 2$, we have $r = 1$ and $r\overline{T}_m = \overline{T}_m = \overline{T}_a$. In this case, $(\overline{T}_m, V)$ is a complete set of sufficient statistics for (μ, λ); so the UMVUE of the failure time distribution function $F(t)$ is $E(I(T_{1m} \leq t)|\overline{T}_m, V)$ by Rao-Blackwell Theorem [12], where

$$E(I(T_{1m} \leq t)|\bar{t}_m, v) = \int_{L_m}^{t} f_{T_{1m}}(z|\bar{t}_m, v)\,dt_{1m} = \int_{L_m}^{t} K(\bar{t}_m, v) z^{-3/2}(n\bar{t}_m - z)^{-3/2} \times \left[v - \frac{n(\bar{t}_m - z)^2}{\bar{t}_m z(n\bar{t}_m - z)} \right]^{nm/2-2} dz,$$
$$\text{for } L_m < t < U_m, \quad (15)$$

with

$$K(\bar{t}_m, v) = \frac{\sqrt{n}(n-1)}{B(nm/2-1, 1/2)\bar{t}_m^{-3/2} v^{(nm-1)/2-1}}, \quad (16)$$

$$L_m = L_m(\bar{t}_m, v) = \frac{\bar{t}_m}{2(n+v\bar{t}_m)}\left\{ n(2+v\bar{t}_m) - \sqrt{4n(n-1)v\bar{t}_m + n^2v^2\bar{t}_m^2} \right\}, \quad (17)$$

$$U_m = U_m(\bar{t}_m, v) = \frac{\bar{t}_m}{2(n+v\bar{t}_m)}\left\{ n(2+v\bar{t}_m) + \sqrt{4n(n-1)v\bar{t}_m + n^2v^2\bar{t}_m^2} \right\}. \quad (18)$$

Proof of (15) is an extension of that for the case with only failure data (see [2]) and is omitted here.

4.3. Estimators of *F(t)* with Only Intermediate Data

In this section, we consider three estimators of $F(t)$ when we have only intermediate, non-failure data. The three estimators are the MLE, an MMLE (Section 4.3.1), and our proposed estimator (Section 4.3.2).

4.3.1. The MLE and an MMLE of $F(t)$

The MLE of $F(t)$, denoted by $\hat{F}_{\text{MLE}}(t)$, is obtained from (4) with μ and λ replaced by their respective MLEs; namely, $r\overline{T}_m$ and r^2nm/V, given in Proposition 4.1. We may also use the UMVUE of $1/\lambda$, $r^{-2}V/(nm-1)$, to obtain an MMLE:

$$\hat{F}_{\text{MMLE}}(t) = \Phi\left[\sqrt{\frac{r^2(nm-1)}{tV}} \left(\frac{t}{r\overline{T}_m} - 1 \right) \right] + e^{\frac{2r^2(nm-1)}{r\overline{T}_m V}} \times \Phi\left[-\sqrt{\frac{r^2(nm-1)}{tV}} \left(\frac{t}{r\overline{T}_m} + 1 \right) \right]. \quad (19)$$

Now, suppose we test kn units up to $a_m (a_m < a)$ to obtain only intermediate data and compute from (19) the corresponding MMLE, denoted $\hat{F}_{\text{MMLE},nk,a_m}(t)$. On the other hand, if we were to test n units until they all failed (i.e., $a_m = a$ and hence $r = 1$), then we can compute the corresponding

MMLE, which will be denoted by $\hat{F}_{\mathrm{MMLE},n,a}(t)$. It is shown in Appendix B that $\hat{F}_{\mathrm{MMLE},nk,a_m}(t)$ can achieve the same levels of precision, up to $O(n^{-1})$, as $\hat{F}_{\mathrm{MMLE},n,a}(t)$ in the sense that

$$\mathrm{Var}(\hat{F}_{\mathrm{MMLE},n,a}(t)) \geq \mathrm{Var}(\hat{F}_{\mathrm{MMLE},nk,a_m}(t)) \tag{20}$$

provided $k \geq r(= a/a_m)$. The conclusion also holds when accuracy [viz. mean squared errors (MSEs)] of the two estimators are compared, since MSE = variance + $(bias)^2$ and $(bias)^2 = (E(\hat{F}(t)) - F(t))^2$ are of $O(n^{-2})$ for both estimators, see (B.2).

4.3.2. *A Third Estimator of $F(t)$*

When $a_m < a$, the failure time T_{1a} for unit 1 can be written as the sum of two independent components:

$$T_{1a} = T_{1m} + (T_{1a} - T_{1m}), \tag{21}$$

where T_{1a} follows $IG(\mu, \lambda)$; T_{1m} and $T_{1a} - T_{1m}$ independently follow $\mathrm{IG}(r^{-1}\mu, r^{-2}\lambda)$ and $\mathrm{IG}(\mu_0, \lambda_0)$, respectively, with $\mu_0 \equiv (a - a_m)/\eta = (1 - 1/r)\mu$ and $\lambda_0 \equiv (a - a_m)^2/\sigma^2 = (1 - 1/r)^2\lambda$.

To estimate $F(t)$, we partitioned the intermediate data $\{T_{ij} | i = 1, \ldots, n, j = 1, \ldots, m\}$ into $\{T_{ij} | i = 1, \ldots, k, j = 1, \ldots, m\}$ and $\{T_{ij} | i = k+1, \ldots, n, j = 1, \ldots, m\}$, where $2 \leq k \leq n - 1$. Then, the cdf of T_{1m} in is estimated by $E(1(T_{1m} \leq t) | \overline{T}_m^{(1)}, V^{(1)})$, which is basically (15) except the values of its $(\overline{T}_m, V)$ are now replaced by $(\overline{T}_m^{(1)}, \overline{V}^{(1)})$ computed from the first subset $\{T_{ij} | i = 1, \ldots, k, j = 1, \ldots, m\}$ (using Proposition 4.1). The unbiased estimator of the pdf of T_{1m} is then

$$\hat{f}(t | \overline{T}_m^{(1)}, V^{(1)}) = \frac{d}{dt} E\big(I(T_{1m} \leq t) | \overline{T}_m^{(1)}, V^{(1)}\big), \tag{22}$$

which has a simple form [as a derivative of (15)].

From the second subset $\{T_{ij} | j = 1, \ldots, m, \text{and } i = k+1, \ldots, n\}$, we first compute $\overline{T}_m^{(2)}$ and $V^{(2)}$ using Proposition 4.1. Then, the unbiased estimators of μ_0 and $1/\lambda_0$ are $\hat{\mu}_0 = (1 - 1/r)(r\overline{T}_m^{(2)})$ and $\widehat{1/\lambda_0} = (1 - 1/r)^{-2}(r^{-2}V^{(2)}/((n-k)m - 1)$, respectively. The MMLE of the cdf of $T_{1a} - T_{1m}$ in (21) is then obtained by replacing μ_0 and $1/\lambda_0$ in the cdf of $\mathrm{IG}(\mu_0, \lambda_0)$ with $\hat{\mu}_0$ and $\widehat{1/\lambda_0}$, respectively, which is $h_t(\hat{\mu}_0, \widehat{1/\lambda_0})$ with h_t defined in (B.1) of Appendix B.

Our proposed estimator of the distribution function $F(t)$ of T_{1a} is the convolution of the two estimated distribution functions of T_{1m} and $T_{1a} - T_{1m}$:

$$\hat{\hat{F}}\big(t | \overline{T}_m^{(1)}, V^{(1)}, \overline{T}_m^{(2)}, V^{(2)}\big) = \int_{L_m^{(1)}}^{\min(t, U_m^{(1)})} h_{t-z}(\hat{\mu}_0, \widehat{1/\lambda_0}) \times \hat{f}\big(z | \overline{T}_m^{(1)}, V^{(1)}\big) dz, \quad \text{for } t > L_m^{(1)}. \tag{23}$$

For highly reliable products, the mean lifetime μ is typically large. Furthermore, for a reasonable product, the variance of its lifetime, $\mathrm{Var}(T_a) = \mu^3/\lambda$, should be relatively small (this can be achieved with robust designs and good statistical process control). Hence, λ should be large (for our example in Section 6, $\lambda = 19221$). Then, from (C.2) in Appendix C, the first-order bias of our proposed estimator in (23) is approximately $\left(1 - \frac{1}{r}\right)^2 \frac{1}{2n} \left(\frac{\partial^2 h_t(\mu, 1/\lambda)}{\partial x_1^2}\right) \frac{r\mu^3}{\lambda}$, whereas the first-order bias of $\hat{F}_{\mathrm{MMLE}}(t)$ in (19) is approximately $\frac{1}{2n} \left(\frac{\partial^2 h_t(\mu, 1/\lambda)}{\partial x_1^2}\right) \frac{r\mu^3}{\lambda}$. Because $0 < 1 - \frac{1}{r} < 1$ with probability 1, our proposed estimator $\hat{\hat{F}}(t | \overline{T}_m^{(1)}, V^{(1)}, \overline{T}_m^{(2)}, V^{(2)})$ has a smaller first-order bias than $\hat{F}_{\mathrm{MMLE}}(t)$ and $\hat{F}_{\mathrm{MLE}}(t)$, approximately.

4.4. Approximate Confidence Intervals for *F(t)* and Its Percentiles

We will first obtain the confidence interval for the lifetime distribution function $F(t)$. Because $\overline{T}_m$ and V are asymptotically normally distributed, by the Delta method [12], both $\hat{F}_{\mathrm{MMLE}}(t)$ and $\hat{\hat{F}}(t | \overline{T}_m^{(1)}, V^{(1)}, \overline{T}_m^{(2)}, V^{(2)})$ are also asymptotically normally distributed. Let $\hat{F}(t)$ denote either of these two estimators; then, for each t

$$\sqrt{n}[\hat{F}(t) - F(t)] \to N\big(0, \sigma_t^2\big) \text{ in distribution, as } n \to \infty, \tag{24}$$

where σ_t^2 is the asymptotic variance of $\sqrt{n}\hat{F}(t)$. For example, if $\hat{F}(t)$ is $\hat{F}_{\mathrm{MMLE}}(t)$ in (19), then $\sigma_t^2 = \left(\frac{\partial h_t(\mu, 1/\lambda)}{\partial x_1}\right)^2 \frac{r\mu^3}{\lambda} + \left(\frac{\partial h_t(\mu, 1/\lambda)}{\partial x_2}\right)^2 \frac{2}{m\lambda^2}$. We may replace the unknown parameters in σ_t^2 with their respective consistent estimates to obtain $\hat{\sigma}_t^2$. Then, from (24), an approximate $100(1 - \alpha)\%$ confidence interval for $F(t)$ at any given t is

$$(\hat{F}(t) - z_{a/2}\hat{\sigma}_t/\sqrt{n}, \hat{F}(t) + z_{a/2}\hat{\sigma}_t/\sqrt{n}), \tag{25}$$

where $z_{a/2}$ is the upper $\alpha/2$ percentage point of the standard normal distribution.

In a reliability analysis, we are also interested in the percentiles of the lifetime distribution. Let t_p be the pth percentile of $F(t)$, that is, $F(t_p) = p (0 \leq p \leq 1)$. An estimate of t_p, denoted by $\hat{t}_p$, is obtained from solving $\hat{F}(\hat{t}_p) = p$. Then, in view of (25), we have

$$\begin{aligned} 1-\alpha &\approx P\{\hat{F}(t_p) - z_{\alpha/2}\hat{\sigma}_{t_p}/\sqrt{n} < F(t_p) < \hat{F}(t_p) + z_{\alpha/2}\hat{\sigma}_{t_p}/\sqrt{n}\} \\ &= P\{F^{-1}[\hat{F}(t_p) - z_{\alpha/2}\hat{\sigma}_{t_p}/\sqrt{n}] < t_p < F^{-1}[\hat{F}(t_p) + z_{\alpha/2}\hat{\sigma}_{t_p}/\sqrt{n}]\} \\ &\approx P\{\hat{F}^{-1}[\hat{F}(\hat{t}_p) - z_{\alpha/2}\hat{\sigma}_{\hat{t}_p}/\sqrt{n}] < t_p < \hat{F}^{-1}[\hat{F}(\hat{t}_p) + z_{\alpha/2}\hat{\sigma}_{\hat{t}_p}/\sqrt{n}]\} \\ &= P\{\hat{F}^{-1}[p - z_{\alpha/2}\hat{\sigma}_{\hat{t}_p}/\sqrt{n}] < t_p < \hat{F}^{-1}[p + z_{\alpha/2}\hat{\sigma}_{\hat{t}_p}/\sqrt{n}]\}, \end{aligned} \tag{26}$$

where the last probability statement gives an approximate $100(1-\alpha)\%$ confidence interval for the pth percentile of the lifetime distribution.

Instead of $F(t)$, one may be interested in the survival function, $S(t)$. Because $S(t) = \Pr(T_a \geq t) = 1 - F(t)$, all our results for $F(t)$ can be modified for $S(t)$. For example, if (a_α, b_α) is a $100(1-\alpha)\%$ confidence interval for $F(t)$, then a confidence interval with the same confidence level for $S(t)$ is $(1-b_\alpha, 1-a_\alpha)$. Furthermore, if s_p is the time that a unit will survive with probability p, then $s_p = t_{1-p}$ and its confidence interval can be obtained using (26).

5. AN EXAMPLE

In Section 3, we proposed a new method of collecting intermediate, nonfailure data from a Wiener degradation or performance process, and then, in Section 4, analytically proved the efficiencies of our estimators for the mean lifetime and the lifetime distribution function. To illustrate the proposed method and the computation required to use our estimators, we consider the contact image sensor (CIS) example from Tseng et al. [24]. A CIS module is a contact-type image-sensing module that consists of a line of LED lamps. A CIS can be used in a fax machine, document scanner, copy machine, mark reader, and other office automation equipment. Because a CIS is designed to be highly reliable (its expected lifetime is measured in years), it is unlikely that it will fail early under normal operating conditions.

The light intensity of an LED lamp has been used as a performance variable–a surrogate of degradation–and an LED is defined as “fail” when its light intensity reaches a constant threshold ([13] and references therein). Instead of a regression model with an independent random error term, Tseng et al. [24] and Liao and Elsayed [13] propose the following correlated process for the degradation of the light intensity:

$$\begin{aligned} L(t_0) &= \exp\{-W(\tau(t_0))\} \\ &= \exp\{-[\eta\tau(t_0) + \sigma B(\tau(t_0))]\}, \quad t_0 \geq 0, \end{aligned} \tag{27}$$

where $\tau(t_0) = t_0^\delta$. Note that $E(L(t_0)) = \exp(-\eta t_0^\delta)$ is the regression function of the regression model in Tseng and Yu [25]. A commonly used threshold for $L(t_0)$ is 0.5; so, the lifetime of a unit is $T_0 = \inf\{t_0 \geq 0 | L(t_0) \leq 0.5\}$. This implies that failure occurs when $W(\tau(t_0))$ passes the transformed threshold $a = -\ell n(0.5) = 0.6932$. If we transform the original time t_0 to $t = \tau(t_0) = t_0^\delta$, then $-\ell n L(t^{1/\delta})$ is the Wiener process $W(t)$ in (1) and the lifetime in the transformed time scale is given in (2).

The original data on $L(t_0)$ were obtained from a consulting project for one of the leading manufacturers of LED lamps in Taiwan and has been used in several other studies. However, the sample size was only 24, of which 18 were with complete failure times. As we will soon see (in Figs. 1a–1c), to obtain reasonably good results, this current sample size may be too small (because we need a larger sample size in our new method of using only intermediate data). Because it has been empirically shown in Liao and Elsayed [13] (using data collected at predetermined time points in an ADT) that the process in (1) is a reasonable one for describing the degradation of the transformed light intensity of an LED, we will use the following MLEs obtained by Tseng et al. [24] to simulate a performance process up to the last nonfailure threshold in our illustration: $\hat{\eta} = 0.0175$, $\hat{\sigma} = 0.005$, and $\hat{\delta} = 0.37$. These estimates will be treated as the true values (because true values were never known) to assess our results based on different sample sizes.

Assume $m = 10$ and the nonfailure threshold are $a_1 = 0.01$, $a_2 = 0.02, \ldots, a_{10} = 0.10 (< a = 0.6932)$. We choose $a_{10} = 0.10$ in our study mainly because it corresponds approximately to 168 h (in the original time scale), which is the company’s current burn-in time under 125°C, 7 eV. Using the intermediate data, we compute $\hat{F}_{\text{MMLE}}(t)$, $\hat{\hat{F}}(t|\overline{T}_m^{(1)}, V^{(1)}, \overline{T}_m^{(2)}, V^{(2)})$ and 95% point-wise confidence intervals for $F(t)$, for $n = 10, 30, 50$, and 100 (Figs. 1a–1d). The results are consistent with our analyses in Section 4. Recall that we have $n = 18$ degradation sample paths with actual failure times at $a = 0.6931$. Because we use the nonfailure thresholds with the last one at $a_m = 0.100$ to collect only intermediate data then, according to the analytical results in Section 4, we need to have a sample size of $(a/a_m)n$ which is about 125, to make our estimates at least equally efficient as the estimates based these 18 degradation sample paths with actual failure times. We demonstrate in Fig. 1d that a sample size of 100 already gives us reasonable confidence bands for the lifetime distribution function. The “true” IG $F(t)$ (with $\mu = a/\hat{\eta}$ and $\lambda = a^2/\hat{\sigma}^2$) is also plotted for each $t \geq 0$. Comparisons of the asymptotic accuracies between $\hat{F}_{\text{MMLE}}(t)$ and $\hat{\hat{F}}(t|\overline{T}_m^{(1)}, V^{(1)}, \overline{T}_m^{(2)}, V^{(2)})$ are consistent with our analytic results. Although these two estimates are fairly close to each other in our example, we have seen cases (e.g., in Chiou [3]) in which the simplified version of the proposed

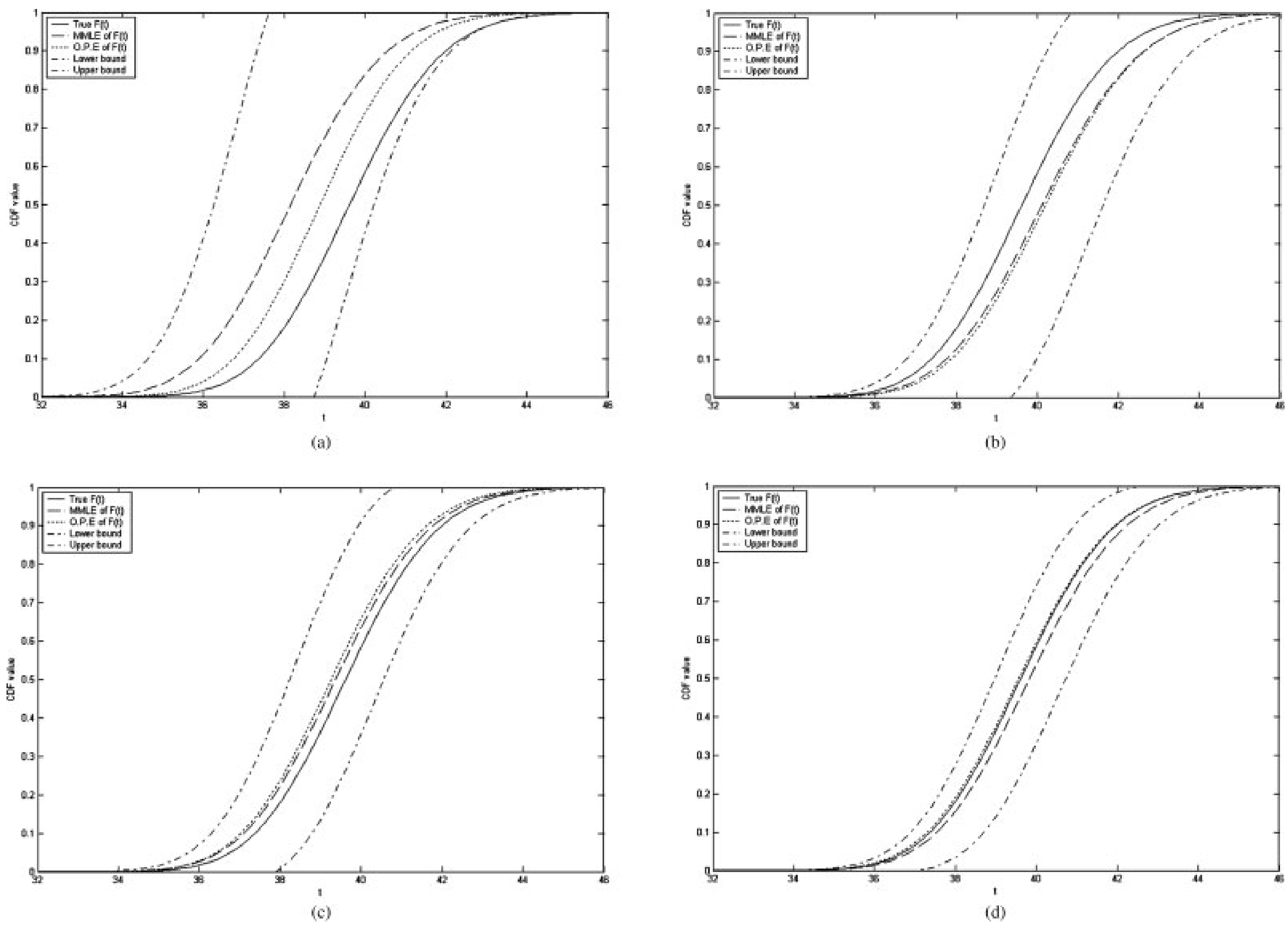

Figure 1. Comparisons of MMLE and our proposed estimate (OPE), and confidence bounds for the failure time distribution function: (a) $n = 10$; (b) $n = 30$; (c) $n = 50$; (d) $n = 100$.

$\hat{\hat{F}}(t|\overline{T}_m^{(1)}, V^{(1)}, \overline{T}_m^{(2)}, V^{(2)})$'s are much closer to the "true" value than $\hat{F}_{\text{MMLE}}(t)$'s. The 95% point-wise confidence intervals for the pth ($0 \le p \le 100\%$) percentiles of the failure time distribution are given in Fig. 2. Because $P(T_0 \le t_0) = P(T^{1/\hat{\delta}} \le t_0) = P(T \le t_o^{\hat{\delta}})$, the pth percentile of the lifetime distribution in the original time scale is estimated by $\hat{t}_p^{1/\hat{\delta}}$, and its approximate confidence interval can be obtained by taking $(1/\hat{\delta})$th roots of the endpoints of the confidence interval given in (26).

As to the computational complexities of our estimators, we first note that values of $\overline{T}_m$ and V are easy to compute using Proposition 4.1. Then, with these estimates, the MMLE of $F(t)$ in (19) is readily obtained. For our proposed estimate $\hat{\hat{F}}$ in (23), we have explained in Section 4.3.2 that it is straightforward to obtain $h_{t-z}(\cdot, \cdot)$ and $\hat{f}(\cdot|\cdot, \cdot)$ when we constructed $\hat{\hat{F}}$. Then, this $\hat{\hat{F}}$ is an integration of the product (or a convolution) of these two functions. Various software programs are available for computing integrals; for example, MATLAB. Hence, the MMLE in (19) and our proposed new estimates in (23) (as functions of t) can be computed easily, and in fact they were computed repeatedly for $t \ge 0$ to obtain the confidence bands in Figs. 1a–1d. The inverse function of F_{MMLE} was also computed repeatedly to obtain the confidence interval for the pth percentile, $0 \le p \le 1$, in Fig. 2.

Another advantage of having intermediate data is that we can check the key Wiener assumption (1) during the test, by testing the goodness-of-fit (GOF) of the IG distribution to the intermediate data. The null hypothesis at each a_j is "H_{0j}: $T_{1j}, T_{2j}, \ldots, T_{nj}$ is distributed as $F_{0j} = IG(\mu_{i0}, \lambda_{j0})$." The unspecified distribution function F_{0j} in H_{0j} will be estimated by its MLE, $\hat{F}_{0j} = IG(\hat{\mu}_{j0}, \hat{\lambda}_{j0})$, where $\hat{\mu}_{j0}$ and $\hat{\lambda}_{j0}$ are the MLEs obtained from Proposition 4.1 using all intermediate data up to a_j.

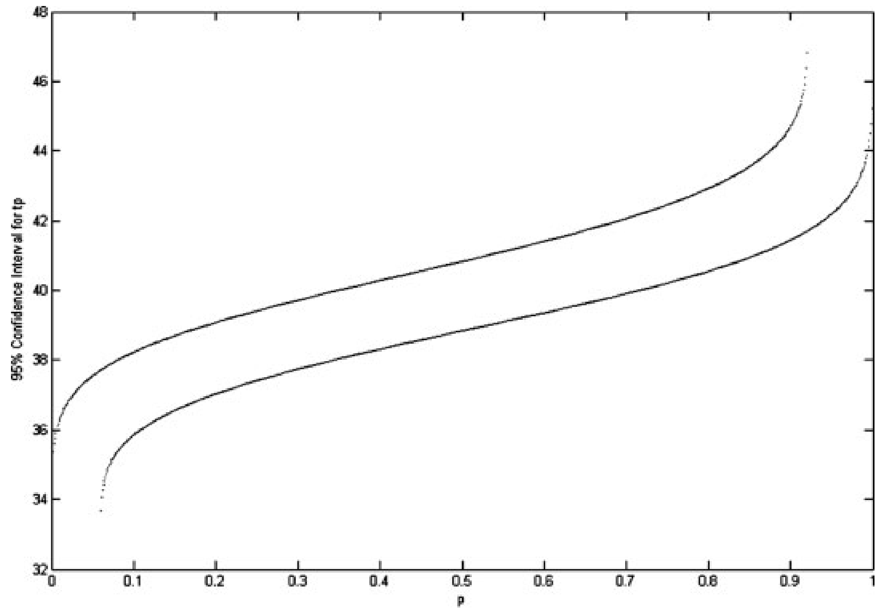

Figure 2. 95% pointwise confidence intervals for percentiles of the failure time distribution (for $n = 100$).

The testing procedures described in Pavur et al. [18] are used for testing H_{0j} at each a_j $(j = 1, \ldots, m)$. For each j, let

$$Z_{ij} = \hat{F}_{0j}(T_{ij}), \quad i = 1, \ldots, n, \tag{28}$$

and the corresponding order statistics be $Z_{(1)j} \leq Z_{(2)j} \leq \ldots \leq Z_{(n)j}$. Then, Anderson-Darling, Cramer-von Mises, and Watson test statistics for H_{0j} are defined, respectively

$$A_j^2 = -n - (1/n)\sum_{i=1}^{n}[(2i-1)\ell n(\mathrm{Z}_{(i)j}) + (2n+1-2i)\ell n(1 - Z_{(i)j})], \tag{29}$$

$$W_j^2 = \sum_{i=1}^{n}[Z_{(i)j} - (2(i)-1)/(2n)]^2 + 1/(12n), \tag{30}$$

$$U_j^2 = W_j^2 - n(\overline{Z}_j - 0.5)^2, \quad \text{with } \overline{Z}_j = \sum_{i=1}^{n} Z_{ij}/n. \tag{31}$$

Let $D_{n,j}$ denote any of these three statistics and

$$D_{n,j}^* = D_{n,j}[\sqrt{n} + \hat{\tau}_1(1/\sqrt{n})] + \hat{\tau}_2(1/n), \tag{32}$$

where $\hat{\tau}_1$ and $\hat{\tau}_2$, which depend on the estimated shape parameter $\hat{\phi}_j = \hat{\lambda}_{j0}/\hat{\mu}_{j0}$ of $\hat{F}_{0j}$, can be found in Pavur et al. [[18], Tables 1–3]. (Note that, while the null distribution of $F_{0,j}(T_{ij})$ does not depend on the shape parameter, the null distribution of $Z_{ij} = \hat{F}_{0j}(T_{ij})$ does.) Reject H_0 if the absolute value of $D_{n,j}^*$ is greater than the absolute value of $\hat{\beta}_\alpha$ where α is a predetermined significance level. The critical values, $\hat{\beta}_\alpha$, for these three tests can be found in [18, Tables 1–3]. Interpolation may be needed if the corresponding $\hat{\phi}_j$ is not listed in the tables. The results of the GOF tests are given in Fig. 3. The horizontal axis is the axis for the predescribed thresholds $a_1, \ldots, a_{10}$. At each, a GOF test is conducted. As indicated, the test statistics do not exceed their respective limits (critical values); so, we can conclude that there is not enough evidence to suggest that the intermediate data do not come from an IG distribution. Our earlier analyses and estimates based on the Wiener assumption (1) are reasonable.

6. CONCLUDING REMARKS

Traditional statistical tools are available for estimating the lifetime distribution function and its parameters when one has failure time data. Instead, we propose to obtain the first-passage times of the test units over certain predetermined nonfailure thresholds during the early stage of a DT. A Wiener process was assumed for tracking the degradation or performance of the product. We then prove the certain efficiency property of our proposed estimator, based only on intermediate data.

To determine the thresholds at which intermediate data are to be collected, we may consider the trade-off between expected total cost of the experiment and (asymptotic) variance of the resulting estimators of the failure time distribution and its percentiles, when designing a degradation experiment. Finally, even with a small a_m, it is still possible that some of

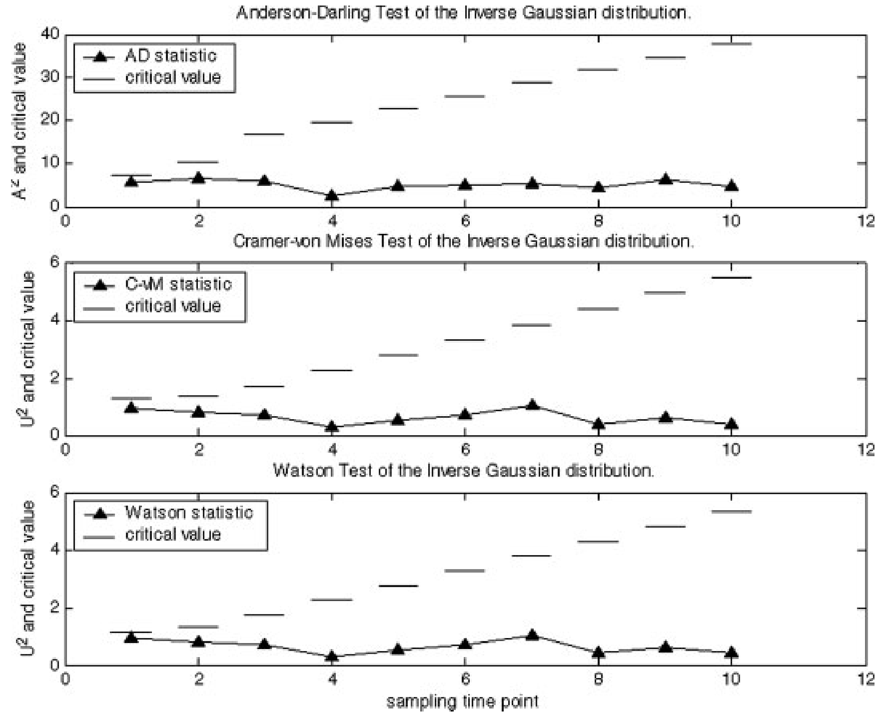

Figure 3. Three charts for verifying IG/Wiener assumption over time.

the test units may not fail. Although we can further lower a_m, we need to further increase the sample size to make the resulting estimators as efficient as the estimators from actual failure time data (as described in Section 4). In cases where testing a large number of units in a DT is not feasible, a possible solution and also a direction for future study is to consider time censoring.

APPENDIX A. PROOF OF PROPOSITION 4.1

The results in this appendix extend Tweedie [26], which dealt with cases with only actual failure time data. The distribution of $\overline{D} \equiv \sum_{i=1}^{n}\sum_{j=1}^{m} D_{ij} = \sum_{i=1}^{n} T_{im}/n \equiv \overline{T}_m$ follows immediately from the fact that T_{im}, $i = 1, \ldots, n$, are independently identically distributed as $IG(a_m/\eta, a_m^2/\sigma^2)$. Furthermore, $r\overline{T}_m = r\overline{D}$ is the UMVUE of μ, because it is unbiased and a function of the complete sufficient statistic $\overline{D}$. Note that the likelihood function of $D_{ij} = d_{ij}$ is given in (11), whose exponent can be written as

$$-\frac{r^{-2}\lambda}{2}\sum_{i=1}^{n}\sum_{j=1}^{k}\left(\frac{r^2}{r_j^2 d_{ij}} - \frac{1}{m\overline{d}}\right) - \frac{n\lambda}{2}\left(\sqrt{\frac{1}{r^2\overline{d}}} - \sqrt{\frac{\overline{d}}{\mu^2}}\right)^2$$
$$= -\frac{\lambda}{2}r^{-2}v - \frac{n\lambda}{2}\left(\sqrt{\frac{1}{r^2\overline{d}}} - \sqrt{\frac{\overline{d}}{\mu^2}}\right)^2. \quad \text{(A1)}$$

Consider the transformations:

$$W_{ij} = D_{ij}, \quad \text{for } j = 1, 2, \ldots, m-1 \quad \text{and} \quad i = 1, 2, \ldots, n,$$
$$W_{im} = D_{im}, \quad \text{for } i = 1, 2, \ldots, n-1, \quad \text{and} \quad W_{nm} = \overline{D}.$$

The Jacobian is n. Then, from (A.1), the joint pdf of W_{ij}'s is

$$n\left(\prod_{j=1}^{m}\frac{1}{2r_j^2\pi}\right)^{\pi/2}\lambda^{nm/2}\left(\prod_{i=1}^{n}\prod_{j=1}^{m-1} w_{ij}^{-3/2}\right)\left(\prod_{i=1}^{n-1} w_{im}^{-3/2}\right)$$
$$\times\left(nw_{nm} - \left(\sum_{i=1}^{n}\sum_{j=1}^{m-1} w_{ij} + \sum_{i=1}^{n-1} w_{im}\right)\right)^{-3/2}$$
$$\times\exp\left\{-\frac{\lambda r^{-2}v}{2}\right\}\exp\left\{-\frac{n\lambda}{2}\left(\sqrt{\frac{1}{r^2 w_{nm}}} - \sqrt{\frac{w_{nm}}{\mu^2}}\right)^2\right\}.$$

Because W_{nm} follows $IG(r^{-1}\mu, nr^{-2}\lambda)$, the conditional pdf of $(W_{11}, \ldots, W_{n-1,m})$, given $W_{nm} = w_{nm}$, can be written as

$$f(w_{11}, \ldots, w_{n-1,m}|w_{nm}) = K\lambda^{(nm-1)/2}e^{-\lambda r^{-2}v/2} \times g(w_{11}, \ldots, w_{n-1,m}, w_{nm}), \quad \text{(A2)}$$

where K is a constant and g is a function of $w_{11}, \ldots, w_{n-1,m}$, and w_{nm}. Note that

$$\iint \ldots \int f(w_{11}, \ldots, w_{n-1,m}|w_{nm})dw_{11}\ldots dw_{n-1,m} = 1, \quad \text{for all } \lambda > 0. \quad \text{(A3)}$$

Now consider the conditional moment generating function of $r^{-2}V$, given $W_{nm} = w_{nm}$:

$$E(e^{t(r^{-2}V)}|W_{nm} = w_{nm}) = \left(\frac{\lambda}{\lambda - 2t}\right)^{(nm-1)/2} \iint \dots \times \int K(\lambda - 2t)^{(nm-1)/2} e^{-(\lambda-2t)r^{-2}v/2} \times g(w_{11}, \dots, w_{n-1,m}, w_{nm}) dw_{11} \dots dw_{n-1,m} = (1 - 2t/\lambda)^{-(nm-1)/2}, \quad \text{for } 0 < t < \lambda/2, \tag{A4}$$

so $E(e^{t(r^{-2}V)}) = E[E(e^{tr^{-2}V}|W_{nm})] = (1 - 2t/\lambda)^{-(nm-1)/2}$. The last equality in (A.4) follows from (A.3). Hence, $(\lambda r^{-2})V$ follows χ^2_{nm-1}. Furthermore, (A.2) or (A.4) proves that $(\lambda r^{-2})V$ or V is independent of $W_{nm} = \overline{D} = \overline{T}_m$ and hence the proposition.

APPENDIX B: PROOF OF (20)

To evaluate $\hat{F}_{\text{MMLE}}(t)$ in (19), let

$$h_t(x_1, x_2) = \Phi\left[\sqrt{\frac{1}{tx_2}}\left(\frac{t}{x_1} - 1\right)\right] + \exp\frac{2}{x_1 x_2} \times \Phi\left[-\sqrt{\frac{1}{tx_2}}\left(\frac{t}{x_1} + 1\right)\right], \tag{B1}$$

then, $\hat{F}_{\text{MMLE}}(t) = h_t(r\overline{T}_m, r^{-2}V/(nm-1))$. Expanding $h_t(r\overline{T}_m, r^{-2}V/(nm-1))$ in a Taylor series about $(\mu, 1/\lambda)$, we can show

$$E(\hat{F}_{\text{MMLE}}(t)) = F(t) + \frac{1}{2}\left\{\left(\frac{\partial^2 h_t(\mu, 1/\lambda)}{\partial x_1^2}\right)\frac{r\mu^3}{n\lambda} + \left(\frac{\partial^2 h_t(\mu, 1/\lambda)}{\partial x_2^2}\right)\frac{2}{(nm-1)\lambda^2}\right\} + O(n^{-2}) = F(t) + \frac{1}{2n}\left\{\left(\frac{\partial^2 h_t(\mu, 1/\lambda)}{\partial x_1^2}\right)\frac{r\mu^3}{\lambda} + \left(\frac{\partial^2 h_t(\mu, 1/\lambda)}{\partial x_2^2}\right)\frac{2}{m\lambda^2}\right\} + O(n^{-2}), \tag{B2}$$

and

$$\text{Var}(\hat{F}_{\text{MMLE}}(t)) = \left\{\left(\frac{\partial h_t(\mu, 1/\lambda)}{dx_1}\right)^2 \frac{r\mu^3}{n\lambda} + \left(\frac{\partial h_t(\mu, 1/\lambda)}{\partial x_2}\right)^2 \frac{2}{(nm-1)\lambda^2}\right\} + O(n^{-2}) = \frac{1}{n}\left\{\left(\frac{\partial h_t(\mu, 1/\lambda)}{\partial x_1}\right)^2 \frac{r\mu^2}{\lambda} + \left(\frac{\partial h_t(\mu, 1/\lambda)}{\partial x_2}\right)^2 \frac{2}{m\lambda^2}\right\} + O(n^{-2}). \tag{B3}$$

so, the MMLE is asymptotically unbiased. Note that the first-order terms (in n) in (B.2) and (B.3) depend on $a_1, \dots, a_m$ through m and r.

We have $\text{Var}(\hat{F}_{\text{MMLE},nk,a_m}(t)) = \frac{1}{nk}\left\{\left(\frac{\partial h_t(\mu,1/\lambda)}{\partial x_1}\right)^2 \frac{r\mu^3}{\lambda} + \left(\frac{\partial h_t(\mu,1/\lambda)}{\partial x_2}\right)^2 \frac{2}{m\lambda^2}\right\}$ and $\text{Var}(\hat{F}_{\text{MMLE},n,a}(t)) = \frac{1}{n}\left\{\left(\frac{\partial h_t(\mu,1/\lambda)}{\partial x_1}\right)^2 \frac{\mu^3}{\lambda} + \left(\frac{\partial h_t(\mu,1/\lambda)}{\partial x_2}\right)^2 \frac{2}{m\lambda^2}\right\}$, up to $O(n^{-1})$. Because $r \geq 1$ and $m \geq 1$, we obtain (20).

APPENDIX C: FIRST-ORDER BIAS OF (23)

Note that

$$E_{\overline{T}_m^{(1)}, V^{(1)}}\left(\int_{L_m^{(1)}}^{\min\left(t, U_m^{(1)}\right)} h_{t-z}(\mu_0, 1/\lambda_0)\hat{f}\left(z|\overline{T}_m^{(1)}, V^{(1)}\right)dz\right) = h_t(\mu, 1/\lambda) = F(t). \tag{C1}$$

Now, we again expand $h_{t-z}(\hat{\mu}_0, \widehat{1/\lambda_0})$ about $(\mu_0, 1/\lambda_0)$ using a Taylor series expansion and, by (C.1) and the independence among $\overline{T}_m^{(1)}, V^{(1)}, \overline{T}_m^{(2)}$, and $V^{(2)}$, we obtain, for fixed k,

$$E\left(\hat{\hat{F}}\left(t|\overline{T}_m^{(1)}, V^{(1)}, \overline{T}_m^{(2)}, V^{(2)}\right)\right) = F(t) + \left(1 - \frac{1}{r}\right)^2 \frac{r\mu^3}{2(n-k)\lambda} \times E\left(\int_{L_m^{(1)}}^{\min\left(t, U_m^{(1)}\right)} \frac{\partial^2 h_{t-z}(\mu_0, 1/\lambda_0)}{\partial x_1^2}\hat{f}\left(z|\overline{T}_m^{(1)}, V^{(1)}\right)dz\right) + \left(1 - \frac{1}{r}\right)^{-4} \frac{1}{[(n-k)m-1]\lambda^2} \times E\left(\int_{L_m^{(1)}}^{\min\left(t, U_m^{(1)}\right)} \frac{\partial^2 h_{t-z}(\mu_0, 1/\lambda_0)}{\partial x_2^2}\hat{f}\left(z|\overline{T}_m^{(1)}, V^{(1)}\right)dz\right) + O(n^{-2}) = F(t) + \left(1 - \frac{1}{r}\right)^2 \frac{1}{2n}\frac{\partial^2 h_t(\mu, 1/\lambda)}{\partial x_1^2}\frac{r\mu^3}{\lambda} + \left(1 - \frac{1}{r}\right)^{-4} \times \frac{1}{2n}\frac{\partial^2 h_t(\mu, 1/\lambda)}{\partial x_2^2}\frac{2}{m\lambda^2} + O(n^{-2}). \tag{C2}$$

from which we obtain the first-order bias of (23).

APPENDIX D: APPROACH TO DEALING WITH RANDOM THRESHOLDS

When a product's actual failure is tracked by an observable (Wiener) degradation process with a fixed but unknown threshold, estimating the failure time distribution and its parameters is possible only when one has complete failure time data. This is because the failure time distribution depends on the threshold a, but only through μ and λ in (5), which can be estimated by the statistics (with $r = 1$) given in Proposition 4.1. If only intermediate data are available, then estimating the lifetime distribution function appears to be impossible because how far to extrapolate the results from intermediate data obtained at a_j's depends on how far a_j's are from a. Nor is it possible for the traditional method of measuring the degradation process at certain prespecified times, because this method gives estimates of the drift parameter η and diffusion coefficient σ^2 in (5), and one needs to know a to estimate the mean and variance of the lifetime distribution.

If the threshold a is random but with a known distribution (e.g., a uniform or beta distribution over a given range), then our results using intermediate data can be considered as conditional results (given a). Recognizing that there will be no data on a, one approach is to first multiply the (conditional) likelihood (given a) in (11) by the known density of a, and then integrate out a to obtain the unconditional likelihood function. This unconditional likelihood (density), however, does not form an exponential family, nor does it have a closed form; hence, estimators of unconditional lifetime parameters such as MLEs or UMUVEs (if they exist) have to be obtained numerically. Consequently, comparisons of various estimators of the lifetime distribution function and its parameters appear to be analytically intractable under this approach.

An alternative is to take an expectation with respect to a to all estimators and parameters containing a. So, we have unbiased estimators, $E(r)\overline{T}_m$ and

$E(r^{-2})V/(nm-1)$, for $E(\mu)$ and $E(1/\lambda)$, respectively, from Proposition 4.1. We can prove $\mathrm{Var}(\overline{T}_a) = E(r)(a_m/n)(\sigma^2/\eta^3)$ for complete data case, and $\mathrm{Var}(E(r)\overline{T}_m) = E^2(r)(a_m/n)(\sigma^2/\eta^3)$ for intermediate data case. Also, we can prove $\mathrm{Var}(E(r^{-2})V/(mn-1)) = [2\sigma^4/(mn-1)][E((1/a^2)^2)]$ for the complete data case, and $\mathrm{Var}(E(r^{-2})V/(mn-1)) = [2\sigma^4/(mn-1)][E(1/a^2)]^2$ for the intermediate data case. Then, we can obtain a similar conclusion as in Section 4.1; that is, the proposed estimator $E(r)\overline{T}_m$ of $E(\mu)$, based on only intermediate data with sample size nk, is at least as efficient as the corresponding estimator $\overline{T}_a$ based on complete failure time data with sample size n, provided $k \geq E(r)$, where $r = a/a_m$ and a_m is fixed (the condition in Section 4.1 was $k \geq r$ when a is known). For the lifetime distribution function, the problem may become that of estimating $F(t|E(\mu), E(1/\lambda))$, where $F(\cdot)$ is given in (4). The estimator in (15) of Section 4.2 for complete failure data case remains unchanged, because a is not directly involved. For the case with only intermediate data in Section 4.3.1, our proposed estimator is $h_t(E(r)\overline{T}_m, E(r^{-2})V/(nm-1))$, with h_t given in (B.1). Following a similar derivation and using the variances of the two estimators given above, we can obtain similar expressions as in (B.2) and (B.3). Then, using the Jensen's inequality (to prove $E((1/a^2)^2) \geq (E(1/a^2))^2$), the conclusion that $\hat{F}_{\mathrm{MMLE},n,a}$ is outperformed by $\hat{F}_{\mathrm{MMLE},nk,a_m}$ in Section 4.3.1 remains valid, but the condition will become $k \geq E(r)$. For Section 4.3.2, our proposed estimator of $F(t|E(\mu), E(1/\lambda))$ is (23), with $\hat{\mu}_0$ and $\widehat{1/\lambda}_0$ replaced by their respective expectations with respect to a. First, equality in (C.2) remains true when we replace all parameters by their respective expectations with respect to a. Then, the conclusion that the proposed estimator in Section 4.3.2 outperforms the MLE and MMLE remains valid, provided $E(r) \geq E((1-1/r)r)$, but this inequality is true because $r \geq 1$ with probability 1.

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