Valuation and Analysis of Collateralized Mortgage Obligations*

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This study develops a model for the valuation of Collateralized Mortgage Obligations (CMOs). The model is based on a two-factor model of the term structure of interest rates and embeds an empirically estimated mortgage prepayment function. The model is used to analyze various CMO tranches, including standard sequential pay fixed-rate tranches, Planned Amortization Class (PAC) tranches, Targeted Amortization Class (TAC) tranches, floating-rate tranches, Interest Only (IO) and Principal Only (PO) tranches, Z-bonds and Residuals. The results of this analysis illustrate the sensitivity of the various tranches to differences in CMO structure, changes in interest rates, the characteristics of the underlying collateral, and mortgage prepayments.
(*Collateralized Mortgage Obligations; CMOS; Mortgage-backed Securities; Tranche*)

1. Introduction

Evolution in the secondary mortgage market has spawned a proliferation of derivative mortgage securities. Predominant among these is the Collateralized Mortgage Obligation (CMO). CMOs are composed of a series of sequential pay bonds or tranches created from a pool of fixed-rate mortgages or generic mortgage backed securities (MBSs). The initial CMOs were relatively simple instruments. Over time, however, the structures of CMOs and CMO tranches have become increasingly complex. In some cases, CMOs have been issued which have as many as 20 different tranches. These can include traditional sequential pay tranches, Planned Amortization Class (PAC) and Targeted Amortization Class (TAC) tranches—all of which may be either fixed or floating rate—Interest Only (IO) and Principal Only (PO) tranches, "super" PO tranches, interest accrual or Z-bonds, and Residuals. The cash flows and, therefore, the values, of the "junior" tranches depend critically upon the structure of the cash flows to the more "senior" tranches and upon the expected pattern of mortgage prepayments. Indeed, the major distinction between MBSs and other default-free government securities is the finite probability of prepayment, or call, of the mortgage collateral which supports them.

Essentially, CMOs redistribute the call risk of the mortgage collateral among investors in the various tranches. Some tranches contain a disproportionate share of this call risk which makes their values extremely sensitive to mortgage prepayment rates. Prepayment rates, in turn, depend mainly upon changes in interest rates: As rates rise, the probability of a mortgage call falls. Depending upon the structure of the tranche and the characteristics of the underlying collateral, however, the call effect and the discounting effect of the interest rate can have either a reinforcing effect or a countervailing effect such that a change in interest rates can either increase or decrease the value of a particular tranche. Indeed, for some CMO structures, the value of a particular tranche may first increase as rates rise and then decline as rates rise further. In many cases, it is not possible to know a priori the exact relation be-

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tween a change in interest rates and the value of the tranche. Rather, it is necessary to model the security directly to take into account the specifics of the CMO structure and the sensitivity of prepayments to changes in interest rates.

This paper develops a model for the valuation of CMOs and examines the effect of mortgage prepayments and changes in interest rates on the value of various CMO tranches. The model builds upon earlier work by Dunn and McConnell (1981); Buser and Hendershott (1984); Kau, Keenan, Muller, and Epperson (1986), and, more particularly, Schwartz and Torous (ST) (1989), all of which develop models for valuing generic MBSs. The primary difference between our model and the models for analyzing generic MBSs is the nature of the stochastic cash flows and terminal conditions of the individual CMO tranches. With CMOs, not only are the cash flows stochastic, but the structure of the tranches and the sequential retirement of their principal balances impose terminal conditions on the valuation equation which are themselves stochastic. Specification of the cash flows to the various tranches in combination with the stochastic terminal conditions is the first major challenge confronted in constructing a CMO valuation model. Specification of these cash flows and terminal conditions leads to a complex set of simultaneous partial differential and algebraic equations. Because this set of equations cannot be solved with conventional finite difference methods, a Monte Carlo simulation procedure is used. Development of the Monte Carlo procedure for the solution of this set of simultaneous equations is the second major challenge confronted in constructing a CMO valuation model.

The next section provides a more detailed description of CMOs. §3 summarizes and recapitulates the ST model for valuing generic MBSs. §4 develops the general CMO valuation procedure and specifies the cash flows and related terminal conditions for various CMO tranches. The Monte Carlo solution procedure is used in §5 to value and analyze several CMO structures. The results of this analysis illustrate the sensitivity of certain CMO tranches to prepayments and changes in interest rates. This analysis also demonstrates that the value and sensitivity of various tranches depend not only upon the CMO structure, but also on the characteristics of the underlying collateral. §6 summarizes and concludes the paper.

2. Collateralized Mortgage Obligations

The first CMO was created by the Federal Home Loan Mortgage Corporation (Freddie Mac) in 1984.[1] The premise behind the creation of the CMO was that the long and uncertain maturities of generic MBSs make them unattractive to some classes of potential investors. The uncertain maturities arise because mortgagors may call, or prepay, their loans at any time without penalty. CMOs were created as a device to induce investment in the mortgage market by these nontraditional mortgage investors. The idea is that by restructuring the cash flows from a pool of mortgages, CMOs can create instruments which cater to specific clienteles of investors with different maturity preferences. To create securities with different maturities, cash flows from mortgage pools or generic MBSs are divided across time. The division of the cash flows across time results in the reallocation of the prepayment risk across the various CMO tranches.

The earliest CMOs were reasonably straightforward and had relatively few tranches. For example, one of the first "plain vanilla" sequential pay CMOs might contain five classes: an *A* tranche, a *B* tranche, a *C* tranche, an interest accrual or "Z-bond", and a Residual. The *A*, *B*, and *C* tranches all have a stated principal amount, a fixed rate of interest, and a stated maturity. The stated maturity is the maximum length of time that would be required to retire the tranche assuming that no mortgagors prepay their loans. In the sequential pay structure, the *A* tranche receives all principal payments, including any prepayments, from the underlying collateral (along with a fixed rate of interest) until its principal is fully retired. During the period that the principal of the *A* tranche is being retired, the holders of the *B* and *C* tranches receive interest payments only. Once the *A* tranche is retired, all principal payments from the

[1] An excellent description of CMOs is available in Edwin H. Duett (1990) and Richard Roll (1987).

underlying collateral are then paid to the *B* tranche until it is fully retired. Principal payments are then made to the *C* tranche.

The interest accrual or Z-bond also has a stated principal balance, a fixed rate of interest, and a stated maturity which is equal to the maturity of the underlying mortgage collateral. However, the Z-bond does not receive any cash flows until the senior tranches are fully retired. Rather, interest payments that would have been paid to the Z-bond are paid to the senior tranches, in order of priority until their principal balances are retired. Concurrently, the principal balance of the Z-bond is increased by the amount of the foregone interest payment. That is, the unpaid interest accrues to the Z-bond. Once the senior tranches are retired, principal payments from the underlying collateral are paid to the Z-bond along with the stated interest rate on the unpaid principal balance. At that time, the accumulated principal balance of the Z-bond should just equal the remaining principal of the underlying collateral (except in those cases in which the CMO is overcollateralized).

The Residual serves as the equity in the CMO. The Residual has no stated face value or interest rate, but receives any cash flows that remain after the cash flow obligations to the other tranches are met each period. Thus, the Residual holders receive cash flows concurrently with the holders of the senior tranches. The Residual cash flows derive from three sources, two of which come about because of structural requirements that CMOs must satisfy: The first structural requirement is that the sum of the face values of the tranches must be less than or equal to the face value of the underlying collateral. To insure that this requirement is satisfied, the CMO may contain "excess" collateral. Any cash flows beyond those promised to the other tranches, either in principal or interest payments, which arise from this overcollateralization are paid to the Residual.

Second, CMOs must be structured so that no rate of prepayments can result in inadequate funds to meet the promised coupon interest payments on any of the tranches. This requirement can be satisfied in either of two ways. The simplest way is that the coupon rate on each tranche is less than or equal to the coupon rate of the underlying collateral. Alternatively, if any tranche has a stated coupon rate greater than that of the underlying collateral, this requirement is met if the sum of the face values of the tranches is less than or equal to the present value of the scheduled principal and interest cash flows of the underlying collateral when discounted at the interest rate of the tranche promising the highest coupon rate. Invariably, satisfaction of these criteria gives rise to excess interest income from the underlying collateral. All excess interest from this source is paid to the Residual.[2]

The third source of cash flow to the Residual derives from reinvestment income. Reinvestment income occurs because of differences in the payment intervals for the CMO tranches and the underlying collateral. For example, mortgages typically have monthly payments, whereas CMOs often have quarterly or semiannual payments. When the CMO is structured, for purposes of allocating the cash flows, some assumption must be made regarding the rate at which funds can be reinvested between payment dates. Typically, this assumption is quite conservative. Any difference between the actual rate earned and the assumed rate is paid to the Residual.

The cash flows to the various tranches depend upon the structure of the CMO, upon the rate at which mortgagors prepay or call their loans, and upon the level of interest rates. The rate at which mortgagors call their loans depends, in turn, upon changes in the level of interest rates. The more junior the tranche, the greater the degree of call risk that it bears. In general, with all else equal, the greater the call risk, the lower the value of the security and the more sensitive is the value of the tranche to changes in interest rates.

Typically, the Residual bears the greatest call risk, followed by the Z-bond, and the sequential pay tranches in descending order of priority. However, even the most senior tranches do contain some uncertainty about the actual maturity of the security. As a way of reducing this uncertainty further, one of the first variations on the sequential pay structure to be introduced was the Planned Amortization Class (PAC) tranche.

In addition to a stated face value and interest rate, a PAC tranche offers a planned amortization schedule

[2] Excess interest income is by far the largest source of Residual cash flow. Thus, most residuals behave like an Interest Only (IO) tranche.

for the retirement of its principal balance. During the planned amortization period, the promised principal payment to the PAC takes precedence over principal payments to the other tranches. At any time preceding the beginning of the planned amortization period, all principal payments from the underlying collateral are allocated to the other tranches according to their seniority. Once the planned amortization period begins, principal payments are made to the PAC according to the planned amortization schedule. Principal payments in excess of the promised amount are then paid to the junior or "support" tranches each period.

Even with a PAC tranche, it is possible that principal payments from the underlying collateral will be "too fast" or "too slow" at some point in time to satisfy the planned amortization schedule. If prepayments are too fast early in the life of the PAC, principal payments will be too low later to meet the amortization schedule. If prepayments are too slow early in the life of the PAC, principal payments will fall below the amount required to satisfy the PAC amortization schedule. Of course, any shortfall in principal payments to the PAC is made up from subsequent principal payments from the collateral.

There is a range of prepayment rates within which the principal payments from the underlying collateral are adequate to meet the principal payments promised to the PAC tranche. This range of prepayments is called the PAC collar. Depending upon the size of the PAC tranche and dollar amount of the underlying collateral, the PAC collar can be designed so as to remove virtually all of the call risk from the PAC tranche. However, this risk does not disappear. It is merely shifted to one of the other tranches. The less the call risk borne by the PAC, the greater the call risk borne by the junior tranches and the more sensitive are their values to changes in prepayments on the underlying collateral.

A Targeted Amortization Class (TAC) tranche is similar to a PAC except that the collar is narrower and, thus, there is a greater probability that principal payments will deviate from the targeted amortization schedule. Hence, TAC tranches bear more call risk than a PAC tranche, but less call risk than the equivalent tranche of a sequential pay CMO.

Although PAC and TAC tranches reduce the investor's uncertainty about the timing of future principal payments, they are still fixed rate securities. As such, the investor is subject to the same interest rate risk as with other fixed rate securities. In an effort to appeal to investors who wish to avoid the interest rate risk inherent in every fixed rate security, CMOs with floating rate tranches were created.[3]

With a floating rate tranche, the interest payment is adjusted periodically and is indexed to a short-term interest rate, usually the 3-month treasury rate or the London Interbank Offer Rate (LIBOR). The interest rate for the floater is usually specified as the index rate plus a predetermined margin. Additionally, the floating rate is often subject to a periodic and / or a lifetime cap which limits the upward movement of the rate over a specified period and / or over the life of the tranche.

Floating rate tranches can be structured to provide either sequential payment or simultaneous payment of principal. The former is like any other sequential pay CMO in which the most senior tranche, which can be the floater, receives all principal payments before the other tranches receive any. With a simultaneous pay floater, principal cash flows from the collateral are divided in fixed proportions between the floating rate tranche and the other tranches. The other tranches can still provide for sequential payment of principal. Also, with the exception of the Residual, the other tranches receive a fixed rate of interest. However, the interest rate of the "support" tranches is set far below the coupon rate of the underlying collateral. These low coupon tranches insure that the interest payments on the floating rate tranche can be made under widely fluctuating interest rate environments.

As with other "low" coupon bonds, the low coupon rate tranches sell at deep discounts from face value, but they do not have the same degree of call protection as do similar "normal" deep discount bonds. Because the interest rate on the underlying collateral is above the rate on the deep discount CMO tranche, if rates fall, the collateral is likely to be called even though the cou-

[3] CMOs with floating rate tranches are backed by fixed rate collateral. They are not adjustable rate mortgage-backed securities of the type analyzed in McConnell and Singh (1991).

pon rate of the tranche is below the current market rate. Thus, the value of a deep discount CMO tranche will be less than the value of an otherwise identical deep discount callable bond.

As with other CMO structures, the risk—in this case, the interest rate risk—avoided by the floater must be absorbed by one of the other tranches. With a floater, the interest rate risk is largely diverted to the Residual. Thus, with a floater, the Residual not only bears considerable call risk, but also an exaggerated interest rate risk. As the index rate increases, the cash flows to the Residual decline dramatically, and vice versa. The Residuals of CMOs with a floating rate tranche are often referred to as "inverse floaters".

As we noted, CMOs can contain tranches with a coupon rate far below the coupon rate of the underlying collateral. In the extreme, a tranche that receives only principal payments can be created. Every CMO that contains one of these Principal Only (PO) tranches also contains an Interest Only (IO) tranche. The IO tranche is structured to receive the interest payments on the principal due to the PO tranche. As soon as the principal of the PO tranche is retired, cash flows to the IO tranche cease. The IO, thus, contains an extreme form of call risk. If interest rates fall and mortgagors call their loans, the IO tranche becomes worthless. Contrarily, as interest rates rise, the probability of mortgage call declines, which increases the probability that the IO will receive income for a longer period of time. Thus, as rates begin to increase, the value of the IO actually increases as well. However, as rates continue to rise, the cash flows are discounted at an increasingly higher rate, such that the value of the IO eventually declines. The value of the PO responds differently to changes in rates. If rates fall and prepayments increase, the PO becomes especially valuable. The principal payments that the investor expected to receive at some distant time in the future arrive much sooner. What were essentially zero coupon bonds with a distant maturity date mature immediately.

A special class of POs embedded in CMOs with a PAC or TAC tranche is known as a "super" PO. Due to the presence of the PAC or TAC tranche, virtually all of the call risk of the underlying collateral is shifted to the other CMO tranches. In this CMO structure, "ordinary" PO tranches become "hyper-sensitive" to mortgagor call decisions. Hence, the term "super" PO.

Finally, as we noted, every CMO with a PO tranche also contains an IO tranche. However, a CMO can contain an IO tranche, even if it has no PO tranche. A regular sequential pay tranche can be created with a coupon rate below the coupon rate of the underlying collateral. The "extra" interest can be used to create an IO tranche. This IO tranche then has many of the characteristics of a Residual.[4]

3. The Schwartz and Torous Model for Valuing Generic MBSs

There are two keys to the valuation of CMOs: (1) a specification of the term structure of interest rates and (2) a model for the valuation of the call risk borne by the various tranches. Schwartz and Torous (ST) (1989) present a mortgage valuation model which embeds an empirical prepayment function for valuing the call option and employs the Brennan and Schwartz (1982) two-factor model of the term structure. We begin with the ST model for the valuation of generic MBSs to construct our model for CMO valuation.

The ST model is based on the assumption that the term structure can be completely specified by the short-term interest rate, r, and the long-term consol rate, l.[5] The dynamics of the two interest rates are modelled as the following stochastic processes:

$$dr = (a_1 + b_1(l - r))dt + \sigma_1 r dz_1,$$
$$dl = \text{p}(a_2 + b_2 l + c_2 r)dt + \sigma_2 l dz_2, \qquad (1)$$

where dz_1 and dz_2 are standard Weiner processes with instantaneous correlation coefficient ρ. Standard arbitrage arguments are then used to generate the following partial differential equation (PDE) which governs the value, M, of every generic MBS:

[4] Very recently, CMOs have been structured which contain no Residual. Rather, the excess interest income that would have been used to create the Residual is used to create an IO tranche.

[5] They further assume no taxes, transactions costs, or shortselling constraints.

$$\tfrac{1}{2}r^2\sigma_1^2M_{rr} + rl\rho\sigma_1\sigma_2M_{rl} + \tfrac{1}{2}l^2\sigma_2^2M_{ll} + (a_1 + b_1(l - r) - \lambda_1\sigma_1 r)M_r + l(\sigma_2^2 + l - r)M_l + \alpha(l - x)M_x - \pi yM_y + M_t + CF(t) - rM = 0, \quad (2)$$

where the subscripts on M represent partial derivatives. The parameter λ_1 is the market premium for short-term interest rate risk. The variable, y, is the fraction of the pool of mortgages surviving until time t with dynamics $dy = -y\pi dt$ and π is the conditional probability of prepayment given that the pool survives until t. The probability of prepayment, π, is empirically estimated and can be specified as a function of any variables that influence mortgagor prepayments. ST specify prepayments to be a function of time, t, the long-term interest rate, l, the history of past interest rates, $x(t)$, and the probability that the pool is outstanding at t, $y(t)$. The variable $x(t)$ is a weighted average of historical mortgage refinancing rates and is defined by

$$x(t) = \alpha \int_{\infty}^{0} \exp(-\alpha r)l(t - s)ds. \quad (3)$$

The dynamics of $x(t)$ are given by

$$dx = \alpha(l - x)dt. \quad (4)$$

This variable captures the effect of past refinancing "cycles" on mortgagors' propensities to refinance.[6]

The term $CF(t)$ characterizes the cash flows to the security being valued. The distinction between the valuation of generic MBSs and the valuation of CMOs lies in the specification of $CF(t)$ and the terminal conditions which M must satisfy. The terminal conditions for CMO tranches depend upon the retirement dates of the principal balances of the various tranches. Because the dates of retirement of the tranches' principal are stochastic, the terminal conditions are also stochastic. The challenge, then, in valuing CMOs is to specify the cash flows to the various tranches and their related terminal conditions.

[6] This variable captures the effect of mortgage "burnout". That is, mortgagors that remain in a pool did not prepay previously when rates were at their current level. Thus, the sensitivity of prepayments is reduced if the pool previously experienced a "low" interest rate cycle. Richard and Roll (1989) expand upon this point.

4. A Model for the Valuation of CMOs

The cash flows to the CMO tranches depend upon the cash flows of the underlying collateral. We assume that the CMO is collateralized by generic MBSs which, in turn, are backed by fixed rate, fully callable, fully amortizing mortgages. Let $F_G(0)$ be the face value of the pool of mortgages (and of the MBSs) at origination, let c be the continuous contract rate of interest on the mortgages, and let T be the term to maturity of the mortgages in years. Then the constant total cash flow payout rate on the mortgage pool is

$$A = cF_G(0)/(1 - \exp(-cT)) \quad (5)$$

and the remaining principal at time t is

$$F_G(t) = (A/c)(1 - \exp(-c(T - t))). \quad (6)$$

The issuer of the MBS receives a fee for servicing the underlying pool of mortgages and an insurance fee is paid to the guarantor of the MBS. These fees are a continuous, fixed proportion of the remaining principal balance of the mortgages. If c_s is the rate deducted for the servicing and guarantee fee, the cash flow rate from the generic MBSs which serve as the CMO collateral is

$$CF_G(t) = y(t)A + y(t)\pi(t)F_G(t) - y(t)c_sF_G(t). \quad (7)$$

The first term on the right in (7) is the scheduled principal and interest payment rate on the collateral that survives until t. The second term denotes the cash flow rate from mortgage prepayments. The third term is the servicing and guarantee fee on the collateral that remains outstanding at t.

Valuation of CMO tranches is based on the PDE expressed in (2). To value a particular type of tranche, the continuous cash flow term, $CF(t)$, must be adapted to reflect the characteristics of that type of tranche. The cash flow characteristics of the tranche determine the stochastic terminal conditions which the value of the tranche must satisfy. Although our model and valuation procedure is a general one that can be used to evaluate CMOs with any number of tranches, for purposes of developing the model, we begin with a CMO which contains a Residual and four tranches—an A tranche, a B tranche, a C tranche, and a Z-bond. We first consider a standard sequential pay CMO. We then develop the model for PACs and TACs, floating rate and discount tranches, and, finally, IOs and POs.

In each case, the cash flows to the Residual are computed as the difference between the cash flows to the underlying MBS collateral and the sum of the cash flows to the individual tranches. Thus, from (7)

$$CF_R(t) = CF_G(t) - [CF_A(t) + CF_B(t) + CF_C(t) + CF_Z(t)] \quad (8)$$

where the subscripts denote the Residual, R, the generic MBS collateral, G, and the other CMO tranches.

4.1. Sequential Pay Tranches

To develop the model for the standard sequential pay CMO, let the initial face values of the tranches be $F_A(0)$, $F_B(0)$, $F_C(0)$, and $F_Z(0)$. Each tranche may have a different coupon rate of interest. Denote these rates as c_A, c_B, c_C, and c_Z. Let c_h be the coupon rate of the tranche paying the highest rate of interest where $h = A, B, C$ or Z. Then, with equations (5) and (6), the face value and cash flow requirements which every CMO must satisfy are

$$F'_G(0) = (A/c_h)(1 - \exp(-c_h T)), \quad (9)$$

$$F_A(0) + F_B(0) + F_C(0) + F_Z(0) \le \text{Min}\,[F_G(0), F'_G(0)], \quad (10)$$

where $F'(0)$ is the present value of the scheduled collateral cash flows when discounted with the highest coupon rate of interest of all the CMO tranches. The condition given in equation (12) insures that the interest payments promised to each tranche can be met regardless of the prepayment rate of the underlying collateral.

Each tranche has a stated maturity based on an amortization schedule which assumes that no mortgages in the pool prepay. For our purposes, however, these maturities are irrelevant. The actual maturities are stochastic and depend upon the rate of mortgage prepayments. Let t_1, t_2, t_3, and t_4 denote the times at which the four tranches are actually retired and let t_5 be the time at which the underlying collateral is actually retired. If the CMO is overcollateralized, t_4 may differ from t_5. Otherwise, $t_4 = t_5$. For purposes of developing the model, the cash flows to the four tranches can be allocated into time intervals demarcated by the dates at which retirement of the tranches actually occurs. The first time interval is denoted as $0 \le t \le t_1$, the second is $t_1 < t \le t_2$, the third is $t_2 < t \le t_3$, the fourth is $t_3 < t \le t_4 \le t_5$, and the fifth is $t_4 \le t \le t_5$.

The demarcation of the time intervals and the remaining principal balance of each tranche depend upon the rate at which the underlying collateral is retired. The cash flows, $CF(t)$, to each tranche, at any time t, in turn, depend upon the remaining principal balances of the tranche at time t. Hence, the cash flows and the remaining principal balances of the various tranches are determined from a set of simultaneous equations. The retirement dates are determined from the cumulative principal paid on the underlying collateral. Thus, time t_1 occurs when the following equation is satisfied:

$$F_A(0) = F_G(0) - y(t_1)F_G(t_1) + F_Z(0)(\exp(c_Z t_1) - 1). \quad (11)$$

The first two terms on the right of (11) give the total principal paid on the underlying collateral. The last term gives the accrued interest on the Z-bond that is used to retire the A tranche. Thus, time t_1 occurs when A is fully retired.

To determine the cash flow rate to A at any time t, it is necessary to determine the (growing) principal of the Z-bond at t and the remaining principal of A. At any time prior to t_3 (when the Z-bond begins to receive principal payments), the principal balance of Z is

$$F_Z(t) = F_Z(0)\exp(c_Z t). \quad (12)$$

The remaining principal of A at any time up to and including t_1 is

$$F_A(t) = F_A(0) - [\{F_G(0) - y(t)F_G(t)\} + F_Z(0)\{\exp(c_Z t) - 1\}]. \quad (13)$$

The cash flow rate for A is then

$$CF_A(t) = y(t)[A - cF_G(t)] + \pi(t)y(t)F_G(t) + c_Z F_Z(t) + c_A F_A(t). \quad (14)$$

The first term on the right of (14) represents the scheduled principal payments from the collateral, the second term gives the cash flows from prepayments on the collateral, the third term is the interest accrued on the Z-bond, and the fourth term is the interest paid on the remaining principal of the A tranche. The first three quantities are used to retire the remaining principal balance of A.

The cash flows to the other tranches during the first time interval, $0 \leq t \leq t_1$, are $CF_B(t) = c_B F_B(0)$; $CF_C(t) = c_C F_C(0)$; $CF_Z(t) = 0$; and $CF_R(t) = CF_G(t) - CF_A(t) - CF_B(t) - CF_C(t)$. That is, the cash flow to the A and B tranches is just the promised interest rate times the initial face value, the cash flow to the Z-bond is zero, and the cash flow to the Residual is just the difference between the cash flow to the collateral and the cash flows to the other tranches.

Time t_2 occurs when the following equation is satisfied

$$F_A(0) + F_B(0) = F_G(0) - y(t_2)F_G(t_2) + F_Z(0)[\exp(c_Z t_2) - 1]. \quad (15)$$

During the interval $t_1 < t \leq t_2$, the remaining principal balance on B is

$$F_B(t) = F_B(0) - [\{F_G(0) - y(t)F_G(t)\} + F_Z(0)\{\exp(c_Z t) - 1\} - F_A(0)] \quad (16)$$

and the cash flow rate to B is

$$CF_B(t) = y(t)[A - cF_G(t)] + \pi(t)y(t)F_G(t) + c_Z F_Z(t) + c_B F_B(t). \quad (17)$$

Terms on the right in (17) are analogous to those in (14). The cash flows to the other tranches and the Residual during the second time interval, $t_1 < t \leq t_2$, are $CF_A(t) = CF_Z(t) = 0$; $CF_C(t) = c_C F_C(0)$; and $CF_R(t) = CF_G(t) - CF_B(t) - CF_C(t)$.

Time t_3 occurs when the B tranche is fully retired. That is, when

$$F_A(0) + F_B(0) + F_C(0) = F_G(0) - y(t_3)F_G(t_3) + F_Z(0)[\exp(c_Z t_3) - 1]. \quad (18)$$

During the third interval, $t_2 < t \leq t_3$, the remaining principal on the C tranche is

$$F_C(t) = F_C(0) - [\{F_G(0) - y(t)F_G(t)\} + F_Z(0)\{\exp(c_Z t) - 1\} - F_A(0) - F_B(0)] \quad (19)$$

and the cash flow rate to C is

$$CF_C = y(t)[A - cF_G(t)] + \pi(t)y(t)F_G(t) + c_Z F_Z(t) + c_C F_C(t). \quad (20)$$

During this interval, the cash flows to the A and B tranches and the Z-bond are zero and the cash flow to the Residual is $CF_R(t) = CF_G(t) - CF_C(t)$.

Time t_4 occurs when the Z tranche is fully retired,

$$F_A(0) + F_B(0) + F_C(0) + F_Z(0) = F_G(0) - y(t_4)F_G(t_4). \quad (21)$$

The remaining principal balance of Z at time t, where $t_3 < t \leq t_4$, is

$$F_Z(t) = F_A(0) + F_B(0) + F_C(0) + F_Z(0) - F_G(0) + y(t)F_G(t) \quad (22)$$

and the cash flow rate to Z is

$$CF_Z(t) = y(t)[A - cF_G(t)] + \pi(t)y(t)F_G(t) + c_Z F_Z(t). \quad (23)$$

The cash flows to the A, B, and C tranches during $t_3 < t \leq t_4$ are zero and $CF_R(t) = CF_G(t) - CF_Z(t)$.

When the CMO is not overcollateralized, $t_4 = t_5$ and the CMO is fully retired at t_4. When the CMO is overcollateralized, t_5 occurs when $y(t_5)F_G(t_5) = 0$. That is, t_5 occurs when the underlying collateral is fully retired. During the interval $t_4 < t \leq t_5$, the cash flow rate to the Residual is $CF_G(t)$ and the cash flows to the other tranches are zero.

Let $M_K(t)$ denote the value of any mortgage security K at time t. The terminal conditions for the solution of the PDEs for the individual securities are, then,

$$M_G(t_5) = M_A(t_1) = M_B(t_2) = M_C(t_3) = M_Z(t_4) = M_R(t_5) = 0. \quad (24)$$

4.2. Planned Amortization Class (PAC) and Targeted Amortization Class (TAC) Tranches

Valuation of a CMO with a PAC tranche requires that the cash flow term of equation (2) incorporate the planned amortization schedule of the PAC. Consider the CMO of the previous section, except now let the C tranche be a PAC. Because the planned amortization schedule of the PAC tranche takes precedence over the principal payments to the A and B tranches, it is necessary to first specify the cash flow term for the PAC. The cash flow terms for the other tranches are based on the difference between the cash flow on the underlying collateral and the cash flow to the PAC.

As before, let t_1, t_2, t_3, and t_4 denote the times at which the four tranches are retired and let t_5 denote the retirement date of the Residual. Let $F_p(0)$ be the face value of the PAC, let T_{p1} and T_{p2} demarcate the beginning and ending dates of the planned amortization schedule, and let $q_P(t)$ be the planned amortization rate of the PAC during the interval T_{p1} through T_{p2}. Let $Q_P(t)$ be the planned cumulative principal payment on the PAC through any time t, given as

$$Q_P(t) = \int_{T_{p1}}^{t} q_P(t)dt, \tag{25}$$

where $Q_P(T_{p1}) = 0$ and $Q_P(T_{p2}) = F_P(0)$, and let $CF_{GPN}(t)$ be the total principal cash flow from the collateral, given as

$$CF_{GPN}(t) = [A - cF_G(t) + \pi(t)F_G(t)]y(t). \tag{26}$$

A deviation of principal payments on the PAC tranche from the planned amortization schedule can occur when prepayment rates on the collateral are extremely high or extremely low. When prepayments are high early in the life of the collateral, the tranches other than the PAC will be retired at a faster rate than anticipated and there will be insufficient cash flow later on to meet the PAC schedule. Contrarily, prepayments can be so low early in the life of the mortgage collateral that no principal payments will be made to any tranche other than the PAC and still the principal payments from the collateral will be insufficient to meet the PAC amortization schedule. If prepayments are extremely rapid, it is possible that all of the support tranches will be retired before the starting date of the planned amortization period of the PAC tranche. In this extreme case, the PAC tranche will begin receiving principal cash flows before T_{p1}. However, for purposes of developing our model, we will ignore this case and assume that principal cash flows to the PAC tranche do not start before T_{p1}.[7]

To model the cash flow rate to the PAC, it is necessary to specify what the cash flows will be when a shortfall occurs. Let $F_P(t)$ denote the remaining principal balance of the PAC at time t, such that the principal already paid is $[F_P(0) - F_P(t)]$. If, due to a low prepayment rate, the total principal paid to the PAC is less than the amount required by the planned amortization schedule, the shortfall is given as

$$S_P(T) = Q_P(t) - [F_P(0) - F_P(t)]. \tag{27}$$

If $I(\cdot)$ denotes the indicator function, which has a value of one if a shortfall occurs and zero otherwise, the total cash flow rate to the PAC at any time t is

$$CF_P(t) = q_P(t) + [CF_{GPN}(t) - q_P(t) + c_Z F_Z(t)] \times I(S(t) > 0) + c_P F_P(t). \tag{28}$$

If CF_{PPN} denotes the principal payment rate to the PAC, then

$$CF_{PPN}(t) = q_P(t) + [CF_{GPN}(t) - q_P(t) + c_Z F_Z(t)] \times I(S(t) > 0). \tag{29}$$

The instant a shortfall takes place, all principal payment from the collateral is diverted to the PAC. At time t, the remaining principal balance on the PAC is

$$F_P(t) = F_P(0) - \int_{T_{p1}}^{t} CF_{PPN}(t)dt. \tag{30}$$

The actual retirement time for the PAC, t_3, occurs when

$$F_P(0) = \int_{T_{p1}}^{t_3} CF_{PPN}(t)dt. \tag{31}$$

The principal payments available for the A and B tranches at any time t prior to the retirement date of the B tranche are equal to the principal payment from the collateral plus the cumulating interest on the Z-bond

[7] We should emphasize that this assumption is made only to simplify the presentation. Our model is general and can incorporate any start date for the retirement of the PAC. In actuality, most PACs are structured so that the probability of principal payments being made to the PAC beginning before T_{p1} are miniscule.

minus the principal payment on the PAC. The other three retirement times are determined as

$$t_1\colon F_G(0) - F_G(t_1)y(t_1) + F_Z(0)[\exp(c_Z t_1) - 1] = F_P(0) - F_P(t_1) + F_A(0), \quad (32)$$

$$t_2\colon F_G(0) - F_G(t_2)y(t_2) + F_Z(0)[\exp(c_Z t_2) - 1] = F_P(0) - F_P(t_2) + F_A(0) + F_B(0), \quad (33)$$

$$t_4\colon F_G(0) - F_G(t_4)y(t_4) = F_P(0) - F_P(t_4) + F_A(0) + F_B(0) + F_Z(0). \quad (34)$$

With a PAC, t_1 occurs before t_2 and t_2 occurs before t_4. However, t_3 may occur anywhere between $t = 0$ and t_5. For purposes of developing the model, we consider only three cases: first, t_3 lies between t_1 and t_2; second, t_3 lies between t_2 and t_4; and third, t_3 is greater than or equal to t_4. That is, we consider the case in which the PAC is retired after the *A* tranche, but before the *B* tranche; the case in which the PAC is retired after the *B* tranche, but before the Z-bond; and the case in which the PAC is retired after the Z-bond.

Time t_1 occurs when equation (32) is satisfied. During the period $0 \le t \le t_1$, the remaining principal balance of the *A* tranche is

$$F_A(t) = F_A(0) + [F_P(0) - F_P(t)] - [F_G(0) - F_G(t)y(t) + F_Z(0)\{\exp(c_Z t) - 1\}], \quad (35)$$

and the cash flow rate to the *A* tranche is

$$CF_A(t) = CF_{GPN}(t) + c_Z F_Z(t) - CF_{PPN}(t) + c_A F_A(t). \quad (36)$$

During this time interval, the cash flows to the PAC are as described in equation (28) and the cash flows to the *B* tranche, the Z-bond and the Residual are the same as with the standard sequential pay CMO described earlier.

Now, consider the case in which $t_2 < t_3 \le t_4$. Time t_2 occurs when equation (33) is satisfied. During the time interval $t_1 < t < t_2$, the *A* tranche receives no cash flows because it has been retired. The cash flow rate to the *B* tranche depends upon its remaining principal balance, which is determined as

$$F_B(t) = F_A(0) + F_B(0) + [F_P(0) - F_P(t)] - [F_G(0) - F_G(t)y(t) + F_Z(0)\{\exp(c_Z t) - 1\}], \quad (37)$$

and the cash flow rate to the *B* tranche is

$$CF_B(t) = CF_{GPN}(t) + c_Z F_Z(t) - CF_{PPN}(t) + c_B F_B(t). \quad (38)$$

The cash flow rate to the PAC is as in equation (28), the cash flow to the Z-bond is zero, and $CF_R(t) = CF_G(t) - CF_B(t) - CF_P(t)$.

Time t_3 occurs when equation (31) is satisfied. During the time interval $t_2 < t \le t_3$, the cash flow to the PAC is as defined in equation (28). The cash flow rate to the Z-bond is

$$CF_Z(t) = CF_{GPN}(t) + c_Z F_Z(T) - CF_{PPN}(t). \quad (39)$$

The cash flows to the *A* and *B* tranche are zero and $CF_R(t) = CF_G(t) - CF_P(t) - CF_Z(t)$.

Time t_4 occurs when the Z-bond is retired, i.e., when equation (34) is satisfied. During the interval $t_3 < t \le t_4$, the remaining principal balance of the Z-bond is

$$F_Z(t) = F_A(0) + F_B(0) + F_P(0) + F_Z(0) - [F_G(0) - F_G(t)y(t)], \quad (40)$$

and the cash flow rate is

$$CF_Z(t) = CF_{GPN}(t) + c_Z F_Z(t). \quad (41)$$

Now consider the case in which $t_1 < t_3 \le t_2$. During the time interval $t_1 < t \le t_3$, the *A* tranche and the Z-bond receive no cash flows. The remaining principal balance and the cash flow rate to the *B* tranche are given by equations (37) and (38), respectively, and the cash flow to the Residual is $CF_R(t) = CF_G(t) - CF_B(t) - CF_P(t)$. During the time interval $t_3 < t \le t_2$, the remaining principal balance on the *B* tranche is given by equation (37) and the cash flow rate is

$$CF_B(t) = CF_{GPN}(t) + c_Z F_Z(t) + c_B F_B(t). \quad (42)$$

Cash flows to the *A* tranche and the Z-bond are zero and $CF_R(t) = CF_G(t) - CF_B(t)$. During the time interval $t_2 < t \le t_4$, cash flows to the *A*, *B*, and PAC tranches are zero and the remaining principal balance and cash flow rate for the Z-bond are given by equations (40) and (41).

Finally, consider the case in which $t_4 < t_3$. This case occurs when prepayments on the collateral are extremely rapid and the *A* and *B* tranches and the Z-bond are all retired before the PAC tranche. This case is similar to that of a CMO with three tranches where the first two are fixed rate sequential pay tranches and the third is a Z-bond with termination date t_4. The cash flow terms to the *A* and *B* tranches and the Z-bond are identical to those for the standard sequential pay CMO except that the principal payments on the PAC, $CF_{PPN}(t)$, are deducted from the principal payments from the collateral to determine the principal cash flows available for the *A* and *B* tranches and the Z-bond. All the principal cash flows which occur after t_4 and until t_3 are paid to the PAC. In this case, the actual retirement date of the PAC occurs after the scheduled retirement date, T_{p2}. Time t_5 occurs when the collateral is fully retired. During the time interval, $\text{Max}\,[t_3, t_4] < t \le t_5$, $CF_R(t) = CF_G(t)$. The terminal conditions for the solution of the PDEs for the individual securities are

$$M_G(t_5) = M_A(t_1) = M_B(t_2) = M_P(t_3) = M_Z(t_4) = M_R(t_5) = 0. \qquad (43)$$

4.3. Floating Rate and Deep Discount Tranches

A CMO with a floating rate tranche may be structured to provide for either sequential or simultaneous retirement of the principal of the various tranches. Despite the differences in the cash flow characteristics of CMOs with and without floating rate tranches, valuation of the tranches in the two types of CMOs is very similar. Since the principal payments on fixed and floating rate tranches are unaffected by their coupon rates of interest, the demarcation of the various time intervals, denoted by t_1, t_2, t_3, t_4 and t_5, is essentially the same as described above for fixed rate sequential pay tranches. Indeed, for CMOs with sequential pay floaters, the demarcation of the various time intervals is identical to those described above for fixed rate sequential pay CMOs. The retirement dates of the sequential pay tranches are determined as before except that the cumulative principal payment to the fixed rate tranches is adjusted downward by the fraction of the principal that is paid to the floater. For CMOs with a simultaneous pay floater, the time intervals are still demarcated by the dates on which the various tranches are retired and the retirement date of the floater coincides with the retirement date of the most junior sequential pay tranche.

Given the retirement dates of the tranches, the only remaining task is to specify their cash flows during each interval. We assume that the interest payments to the floater are tied to an index represented by L with dynamics:

$$dL = (a_3 + b_3(l - L))dt + \sigma_3 L dz_3, \qquad (44)$$

where dz_3 is a standard Weiner process. The coupon rate on the floater is then given as $c_F = L + m$, where m is a fixed margin over the index. The correlation coefficients of dz_3 with dz_1 and dz_2 are ρ_{13} and ρ_{23}, respectively. For a CMO with a sequential pay floater, the cash flow equations are identical to those for a CMO with sequential pay fixed rate tranches except that c_F is substituted for a fixed coupon rate. For a CMO with a simultaneous pay floater, the principal cash flows to the tranches are adjusted to reflect the fixed proportion paid to the floater and the sequential tranches.

4.4. Interest Only (IO) and Principal Only (PO) Tranches

IO and PO tranches are created in a CMO structure by separating the interest and principal cash flows of any tranche and repackaging them as different securities. For instance, in our original example, if the *B* tranche is divided into an IO and a PO, the cash flow rates to the respective tranches are

$$CF_{IO}(t) = c_B F_B(t), \qquad (45)$$

$$CF_{PO}(t) = CF_B(t) - c_B F_B(t). \qquad (46)$$

If the *B* tranche is part of a standard sequential pay CMO structure, the cash flow term and the remaining principal balance term for the *B* tranche are as specified in the derivation of the cash flow equations for a sequential pay CMO. If the CMO contains a PAC tranche, these two terms are as specified in the derivation of cash flow equations for a CMO with a PAC tranche, in which case a "super" PO is created. The terminal conditions are

$$M_{IO}(t_2) = M_{PO}(t_2) = 0. \qquad (47)$$

5. Implementing the CMO Valuation Model

5.1. The Monte Carlo Solution Procedure

To solve the simultaneous PDEs and algebraic equations we use Monte Carlo simulation as outlined in Boyle (1977). Valuation equation (2) contains two state variables, the short-term rate and the long-term rate with dynamics as specified in equation (1). The dynamics of the risk adjusted interest rate process are:

$$dr = (a_1 + b_1(l - r) - \lambda_1 r\sigma_1)dt + r\sigma_1 dz_1,$$
$$dl = l(\sigma_2^2 + l - r)dt + l\sigma_2 dz_2. \quad (48)$$

To implement the Monte Carlo simulation procedure, it is necessary to specify the parameters of the interest rate processes. For the illustrations which follow, we use the parameters estimated by ST (1989). Additionally, it is necessary to specify the cash flows for each state. Doing so requires a state dependent mortgage prepayment function. Again, for illustrative purposes, we employ the empirical prepayment function estimated by ST.[8] The functional form is

$$\pi(t; v, \theta) = \pi_0(t; \gamma, p)\exp(\beta v), \quad (49)$$

where the baseline hazard function π_0 is log-logistic,

$$\pi_0(t; \gamma, p) = \gamma p(\gamma t)^{p-1}/(1 + (\gamma t)^p). \quad (50)$$

The three exogenous variables, $v_1(t)$, $v_2(t)$, and $v_3(t)$, used by ST are: (1) the difference between the collateral coupon rate and the long-term interest rate lagged by three months, $(c - l(t - 3))$, (2) the cube of this term to capture nonlinearity, and (3) the log of the fraction of the pool surviving.[9] The variable x, which represents the history of long-term interest rates in the differential equation, is proxied by $l(t - 3)$.[10]

5.2. Valuing and Analyzing CMOs: Some Illustrations

To illustrate the model, we first consider a CMO with three fixed rate sequential pay tranches, a Z-bond and a Residual. We then introduce PACs, floaters, IOs, and POs. These analyses illustrate the effect of CMO structure on the values of various tranches. We also examine the sensitivity of the values to changes in interest rates, which, in turn, indicates the sensitivity of the values of different tranches to changes in prepayment rates. Finally, we examine the effect of aging of the collateral on the values of the tranches by considering CMOs supported by 30-year collateral with different remaining terms-to-maturity.

In our illustrations, the underlying collateral is assumed to be generic MBSs supported by a pool of newly-issued 30-year fixed-rate mortgages bearing a coupon rate of 9.5 percent. The servicing and guarantee fee is

[8] Dale-Johnson and Langetieg (1987) develop a contingent claims valuation model for CMOs with two fixed rate tranches and a Residual. Their valuation equation contains two stochastic variables, the short-term interest rate and the fraction of the mortgage collateral retired, and assumes, rather than estimates, a linear relation between interest rates and mortgage prepayments.

[9] The effect of refinancing costs on the mortgagors' prepayment decision is captured by the variable, $v_1(t) = c - l(t - s)$. Here, c denotes the coupon rate on the underlying mortgages and l is the consol rate which proxies the current mortgage rate. There is always a time lag between a change in the current mortgage rate and the actual refinancing. Hence, the lagged consol rate is used for this covariate. Schwartz and Torous (1989) find that a value of s equal to three months provides the best fit. The variable $v_2(t) = (c - l(t - s))^3$ is used to capture the effect of accelerated refinancing when the refinancing rate is substantially lower than the mortgagor's contract rate. The effect of the proportion of mortgages previously prepaid in a pool is reflected by the variable $v_3(t) = \ln(A0_t/A0_t^*)$, where $A0_t$ represents the dollar amount of the pool outstanding at time t, and $A0_t^*$ is the total principal amount which would be outstanding in the absence of prepayments.

[10] The maximum likelihood estimates of the prepayment function

$$\pi = \left((\gamma p(\gamma t)^{p-1})\Big/(1 + (\gamma t)^p)\exp\left(\sum_{i=1}^{3}\beta_i v_i\right)\right)$$

obtained by Schwartz and Torous (1989) are provided with the jackknifed standard deviations in parentheses: $\gamma = 0.00572$ (0.00187), $p = 2.35014$ (0.12103), $\beta_1 = 0.39678$ (0.04346), $\beta_2 = 0.00356$ (0.00126), $\beta_3 = 3.74351$ (0.44697). Time, t, is in years and the consol rate, l, is expressed as a percentage. The maximum likelihood estimates of the interest rate processes $dr = (a_1 + b_1(l - r))dt + \sigma_1 r dz_1$ and $dl = (a_2 + b_2 l + c_2 r)dt + \sigma_2 l dz_2$ with $dz_1 dz_2 = \rho dt$ obtained by Schwartz and Torous are provided with standard errors in parentheses: $a_1 = -0.0800$ (0.0359), $b_1 = 0.0382$ (0.0174), $a_2 = -0.0033$ (0.0063), $b_2 = -0.0007$ (0.0019), $c_2 = 0.0008$ (0.0016), $\sigma_1 = 0.0262$, $\sigma_2 = 0.0173$ and $\rho = 0.3732$. Time is expressed in years and interest rates in percentages. The market price of risk is $\lambda_1 = 0.01$. The dynamics of the index are similar to the short-term rate, r. We use the parameters of r for the index and a correlation of 0.9 between r and the index.

assumed to be 0.5 percent per year. For the fixed rate sequential pay CMO, the coupon rates of interest on the *A*, *B* and *C* tranches are assumed to be 8%, 9% and 9%, respectively. The coupon rate of the Z-bond is 9%. The face values of the *A*, *B* and *C* tranches each comprise 30% of the face value of the underlying collateral. The face value of the Z-bond comprises the remaining 10%. The CMO is not overcollateralized. The underlying collateral provides for monthly payments, whereas the CMO is assumed to provide for quarterly payments. In addition to the excess interest, the Residual receives reinvestment income due to the differential timing of the payments on the underlying collateral and the CMO. However, because the model described in §4 is developed in continuous time, in which all cash flows are continuously reinvested at the prevailing short-term rate, the equations cannot provide for reinvestment income due to cash flow timing differences. When the simulations are performed, the model is discretized, which allows reinvestment income to the Residual to be taken into account. In the simulations, the monthly cash flows from the collateral are assumed to be reinvested at the prevailing short-term rate until the next quarterly payment to the CMO tranches.

The values of the various tranches are displayed in Panel A of Table 1. The last column gives the values of the generic MBS. The columns labelled *A*, *B*, *C* and *Z* express the values of the CMO tranches as a percentage of the face value of the relevant tranche. The column labelled *R* gives the value of the Residual as a percentage of the face value of the collateral. Thus, in every case, the value of the collateral is equal to the value of the Residual plus the weighted sum of the values of the other tranches. As shown in Table 1, the collateral and the values of the various tranches are insensitive to wide swings in the short-term interest rate. This turns out to be true for all the structures we analyze. For that reason, in subsequent illustrations, we do not examine the sensitivity of value to changes in the short-term rate.

The table also shows that the value of the generic MBS and the various tranches is highly sensitive to changes in the long-term rate, however, the degree of sensitivity differs across the tranches. In general, the more senior the tranche, the less sensitive is its value to changes in the long-term rate. For example, as the long-term rate increases from 7% to 11%, the value of the Z-bond declines by about 60%, whereas the value of the *A* tranche declines by only 6%. The reason for this is that the more junior tranches have longer maturities and longer maturity securities are much more sensitive to changes in interest rates.

We now consider a CMO with the same structure as above except that the *C* tranche is replaced by a PAC. The planned amortization schedule of the PAC begins at the end of the 24th month and ends at the end of the 114th month. Panel B of Table 1 presents the values of the various tranches for this CMO. A comparison of the two panels of the table reveals that the *A* and *B* tranches of the CMO with the PAC are much more sensitive to changes in interest rates than those of the CMO without the PAC. For example, as the long-term rate drops from 11% to 7%, the *B* tranche in Panel B increases in value by 33%, whereas the *B* tranche in Panel A increases in value by only 21%. This analysis also illustrates that the PAC tranche is much less sensitive to changes in interest rates than the *C* tranche of the sequential pay CMO in Panel A. For example, as the long-term rate falls from 11% to 7%, the *C* tranche increases in value by 34%, whereas the PAC increases in value by only 17%. This result is, of course, a direct consequence of the reduced sensitivity of the PAC cash flows to changes in interest rates. That is, the planned amortization schedule does accomplish its objective. The values of the Z-bond and the Residual are virtually unchanged between the two panels. This occurs because the protection for the PAC comes from the *A* and *B* tranches so the cash flows to the Z-bond and Residual are essentially unchanged.

Table 2 presents the values of a CMO with a floating rate tranche. The interest rate on the floater is equal to the index rate plus a margin of 40 basis points. Panel A presents the values for a sequential pay CMO with the *A* tranche being a floater. The coupon rates on the *B* and *C* tranches and the Z-bond are all 6%. Panel B gives the values for a CMO with sequential pay fixed rate tranches where the *A* tranche has a coupon rate of 9% and the *B* and *C* tranches and the Z-bond have the coupon rate of 6%. The relative face values of the different tranches in Panels A and B of Table 2 are the same as in Table 1.

As with any index linked security, the value of the floater is quite insensitive to changes in interest rates. As the rate drops from 11% to 7%, the value of the floater changes by less than 0.2%. Additionally, a comparison of Panel A with Panel B shows that the values of the *B* and *C* tranches and the Z-bond are unaffected by whether the *A* tranche is fixed or floating rate. These tranches receive the same interest and principal payments under either structure. Where the interesting difference shows up is in the Residual. First, because of the low coupon interest rates of the *B* and *C* tranches, greater interest cash flows are diverted to the Residual than in the example of Table 1. Thus, the Residual has a much higher relative value. Second, for the CMO with

Table 1 Valuation and Analysis of a Standard Sequential Pay CMO and a Sequential Pay CMO with a PAC Tranche

Collateral Attributes: Coupon = 9.5%; Maturity = 360 months; Servicing and Guarantee Fee = 0.5%.

A. Values of sequential pay fixed-rate CMO tranches[a]
Face Value of Tranche/Face Value of Collateral: *A* = 0.30; *B* = 0.30; *C* = 0.30; *Z* = 0.10.
Coupon Rates of Different Tranches: *A* = 8%; *B* = 9%; *C* = 9%; *Z* = 9%.

Short-Term Interest Rate	Long-Term Interest Rate	*A*	*B*	*C*	*Z*	Residual	Generic MBS
7%	7%	101.45	107.35	113.98	146.13	0.37	111.81
7	8	100.07	104.06	106.80	120.03	0.59	105.87
7	9	98.18	99.75	99.31	97.44	0.80	99.71
7	10	95.74	94.60	91.91	78.29	0.99	93.50
7	11	92.71	88.83	84.84	62.29	1.18	87.32
9%	7%	100.97	106.88	113.63	146.37	0.36	111.44
9	8	99.62	103.67	106.54	120.41	0.58	105.57
9	9	97.76	99.44	99.14	97.91	0.79	99.48
9	10	95.38	94.38	91.80	78.79	0.98	93.33
9	11	92.41	88.70	84.78	62.80	1.16	87.21
11%	7%	100.48	106.41	113.28	146.61	0.35	111.07
11	8	99.17	103.28	106.29	120.79	0.57	105.27
11	9	97.35	99.14	98.96	98.37	0.78	99.25
11	10	95.02	94.17	91.70	79.29	0.97	93.16
11	11	92.11	88.57	84.72	63.30	1.15	87.10

B. Values of sequential pay fixed-rate CMO tranches when the CMO includes a PAC tranche[a]
Face Value of Tranche/Face Value of Collateral: *A* = 0.30; *B*= 0.30; PAC = 0.30; *Z* = 0.10.
Coupon Rates of Different Tranches: *A* = 8%; *B* = 9%; PAC = 9%; *Z* = 9%.

Short Term Interest Rate	Long-Term Interest Rate	*A*	*B*	PAC[b]	*Z*	Residual	Generic MBS
9%	7%	101.06	112.73	107.60	146.37	0.39	111.44
9	8	99.58	106.44	103.49	120.42	0.68	105.57
9	9	96.87	99.14	99.58	97.70	1.04	99.48
9	10	92.72	91.81	95.84	78.64	1.36	93.33
9	11	87.26	84.78	92.28	62.77	1.64	87.21

[a] The values of the *A*, *B*, and *C* tranches and the *Z*-bond are expressed as a percentage of their initial face values. The value of the Residual is expressed as a percentage of the initial face value of the collateral.
[b] Start date of PAC amortization schedule = 24 months; End date of PAC amortization schedule = 114 months.

the floater, the Residual value moves in the opposite direction of the interest rate. This illustrates the inverse floating rate nature of the Residual in this structure. As the interest rate falls, the interest cash flows to the Residual increase because the interest cash flows to the floater decline with the consequence that value increases as the interest rate increases.

Panel C presents the results for a CMO with a simultaneous pay floating rate tranche accompanied by three sequential pay fixed rate tranches, a Z-bond, and

Table 2 Valuation and Analysis of CMOs with Floating Rate Tranches

Collateral Attributes: Coupon = 9.5%; Maturity = 360 months; Servicing and Guarantee Fee = 0.5%.

A. Values of sequential pay CMO tranches when the CMO contains a floating rate tranche[a]
Face Value of Tranche/Face Value of Collateral: Floater = 0.30; *B* = 0.30; *C* = 0.30; *Z* = 0.10.
Coupon Rates of Different Tranches: Floater = (Index + 0.4%.); *B* = 6%; *C* = 6%; *Z* = 6%.

Short-Term Interest Rate	Long-Term Interest Rate	Floater[b]	*B*	*C*	*Z*	Residual	Generic MBS
9%	7%	100.76	95.32	90.95	78.67	17.51	111.44
9	8	100.73	90.30	82.62	61.72	17.36	105.57
9	9	100.69	84.33	74.64	47.96	16.84	99.48
9	10	100.64	77.67	67.30	36.96	16.01	93.33
9	11	100.57	70.59	60.71	28.26	14.89	87.21

B. Values of sequential pay fixed-rate CMO tranches[a]
Face Value of Tranche/Face Value of Collateral: *A* = 0.30; *B*= 0.30; *C* = 0.30; *Z* = 0.10.
Coupon Rates of Different Tranches: *A* = 6%; *B* = 6%; *C* = 6%; *Z* = 6%.

Short Term Interest Rate	Long-Term Interest Rate	*A*	*B*	*C*	*Z*	Residual	Generic MBS
9%	7%	102.57	95.32	90.94	78.58	16.96	111.44
9	8	101.44	90.30	82.60	61.63	17.14	105.57
9	9	99.77	84.33	74.62	47.88	17.11	99.48
9	10	97.53	77.67	67.28	36.89	16.94	93.33
9	11	94.66	70.59	60.68	28.19	16.66	87.21

C. Values of simultaneous pay CMO tranches when the CMO contains a floating rate tranche[a]
Face Value of Tranche/Face Value of Collateral: Floater = 0.45; *A* = 0.15; *B*= 0.15; *C* = 0.15; *Z* = 0.10.
Coupon Rates of Different Tranches: Floater = (Index + 0.4%.); *A* = 6%; *B* = 6%; *C* = 6%; *Z* = 6%.

Short Term Interest Rate	Long-Term Interest Rate	Floater[b]	*A*	*B*	*C*	*Z*	Residual	Generic MBS
9%	7%	101.84	97.85	95.32	90.95	78.67	15.15	111.44
9	8	101.78	95.95	90.30	82.62	61.72	13.32	105.57
9	9	101.68	93.51	84.33	74.64	47.96	11.11	99.48
9	10	101.54	90.45	77.67	67.30	36.96	8.68	93.33
9	11	101.22	86.70	70.59	60.71	28.26	6.20	87.21

[a] The values of the *A*, *B*, and *C* tranches and the *Z*-bond are expressed as a percentage of their initial face values. The value of the Residual is expressed as a percentage of the initial face value of the collateral.
[b] The index rate of the floater is the short-term interest rate.

a Residual. The face value of the Z-bond continues to be 10% of the face value of the collateral. The sum of the face values of the other tranches is equal to the remaining 90% of the face value of the collateral. This amount is then split evenly between the floater and the three sequential pay fixed rate tranches. Under the simultaneous pay structure, the floater will be retired on the same date as the C tranche. The interesting comparison here is the Residual of this structure with the Residual in Panel A. In both structures, the Residual exhibits the characteristic of an inverse floater. However, because the floater is outstanding over a much longer time period with the simultaneous pay structure, this Residual is much more sensitive to changes in interest rates. For example, in Panel C, as the long-term rate declines from 11% to 7%, the Residual increases in value by 144%; in Panel A, for the same change in the long-term rate, the Residual increases in value by only 18%.

To analyze CMO structures with IO and PO tranches, the *B* tranche of the two CMOs in Table 1 is split into an interest only and a principal only tranche. Panel A of Table 3 presents the results for the sequential pay CMO with an IO and a PO tranche. Panel B presents the results for the CMO with a PAC and an IO and a PO tranche. In both structures, the PO is highly sensitive to changes in the long-term interest rate. However, the sensitivity is much more pronounced in Panel B, which illustrates the super PO characteristic of the tranche when the PO is accompanied by a PAC. In Panel B, when the CMO includes a PAC, a decline in the long-term rate from 11% to 7%, the PO increases in value by 149%; for the CMO without the PAC (Panel A), the PO increases in value by 32%. This differential effect of interest rate changes on the PO occurs because the "super PO" in Panel B is bearing the call risk diverted from the PAC. Indeed, the PAC tranche (in Panel B)

Table 3 Valuation and Analysis of CMOs with Interest Only and Principal Only Tranches

Collateral Attributes: Coupon = 9.5%; Maturity = 360 months; Servicing and Guarantee Fee = 0.5%.

A. Values of sequential pay CMO tranches when the CMO contains an IO and a PO tranche[a]
Face Value of Tranche/Face Value of Collateral: *A* = 0.30; PO = 0.30; *C* = 0.30; *Z* = 0.10.
Coupon Rates of Different Tranches: *A* = 8%; IO = 9%; *C* = 9%; *Z* = 9%.

Short-Term Interest Rate	Long-Term Interest Rate	*A*	IO	PO	*C*	*Z*	Residual	Generic MBS
9%	7%	100.97	34.01	72.87	113.63	146.37	0.36	111.44
9	8	99.62	38.36	65.31	106.54	120.41	0.58	105.57
9	9	97.76	42.13	57.32	99.14	97.91	0.79	99.48
9	10	95.38	45.30	49.08	91.80	78.79	0.98	93.33
9	11	92.41	47.88	40.83	84.78	62.80	1.16	87.10

B. Values of CMO tranches when the CMO contains a PAC tranche, an IO tranche, and a super PO tranche[a]
Face Value of Tranche/Face Value of Collateral: *A* = 0.30; PO = 0.30; PAC = 0.30; *Z* = 0.10.
Coupon Rates of Different Tranches: *A* = 8%; IO = 9%; PAC = 9%; *Z* = 9%.

Short Term Interest Rate	Long-Term Interest Rate	*A*	IO	Super PO	PAC	*Z*	Residual	Generic MBS
9%	7%	101.06	61.10	51.63	107.60	146.37	0.39	111.44
9	8	99.58	65.56	40.88	103.49	120.42	0.68	105.57
9	9	96.87	66.34	32.80	99.58	97.70	1.04	99.48
9	10	92.72	65.51	26.30	95.84	78.64	1.36	93.33
9	11	87.26	64.03	20.75	92.28	62.77	1.64	87.10

[a] The values of the *A*, *B*, and *C* tranches and the *Z*-bond are expressed as a percentage of their initial face values. The value of the Residual is expressed as a percentage of the initial face value of the collateral.

is much less sensitive to changes in the interest rate than is the C tranche of Panel B. The dampened sensitivity of the PAC relative to the tranche is the mirror image of the heightened sensitivity of the super PO in Panel B.

Table 4 illustrates the effect of aging. This table presents the values of the tranches of the standard sequential pay CMO of Table 1 as time passes. In Table 1, the CMO is newly issued. Panel A of Table 4 gives the values of the tranches of the same CMO after it has

Table 4 Valuation and Analysis of Standard Sequential Pay CMO Backed by Collateral with Varying Remaining Terms to Maturity

Collateral Attributes: Coupon = 9.5%; Servicing and Guarantee Fee = 0.5%.

A. Values of sequential pay fixed-rate CMO tranches with collateral which has 336 months remaining to maturity[a]
Fraction of Original Mortgage Pool Surviving: 0.90
Face Value of Tranche/Face Value of Collateral: A = 0.30; B = 0.30; C = 0.30; Z = 0.10.
Coupon Rates of Different Tranches: A = 8%; B = 9%; C = 9%; Z = 9%.

Short-Term Interest Rate	Long-Term Interest Rate	*A*	*B*	*C*	*Z*	Residual	Generic MBS
9%	7%	100.37	106.31	113.97	145.54	0.15	110.90
9	8	99.67	103.54	106.70	120.11	0.38	105.37
9	9	98.38	99.45	99.16	98.07	0.61	99.51
9	10	96.41	94.35	91.76	79.34	0.83	93.52
9	11	93.67	88.57	84.73	63.64	1.04	87.50

B. Values of sequential pay fixed-rate CMO tranches with collateral which has 312 months remaining to maturity[a]
Fraction of Original Mortgage Pool Surviving: 0.80.
Face Value of Tranche/Face Value of Collateral: A = 0.30; B = 0.30; C = 0.30; Z = 0.10.
Coupon Rates of Different Tranches: A = 8%; B = 9%; C = 9%; Z = 9%.

Short-Term Interest Rate	Long-Term Interest Rate	*A*	*B*	*C*	*Z*	Residual	Generic MBS
9%	7%	100.39	107.17	114.65	144.60	0.20	111.33
9	8	99.67	103.99	106.95	119.72	0.45	105.60
9	9	98.28	99.42	99.18	98.20	0.68	99.57
9	10	96.14	93.96	91.69	79.93	0.91	93.44
9	11	93.20	88.02	84.67	64.63	1.12	87.35

C. Values of sequential pay fixed-rate CMO tranches with collateral which has 288 months remaining to maturity[a]
Fraction of Original Mortgage Pool Surviving: 0.70.
Face Value of Tranche/Face Value of Collateral: A = 0.30; B = 0.30; C = 0.30; Z = 0.10.
Coupon Rates of Different Tranches: A = 8%; B = 9%; C = 9%; Z = 9%.

Short-Term Interest Rate	Long-Term Interest Rate	*A*	*B*	*C*	*Z*	Residual	Generic MBS
9%	7%	100.68	108.78	115.29	143.17	0.38	112.12
9	8	99.64	106.64	107.14	119.03	0.64	105.96
9	9	97.81	99.35	99.15	98.21	0.88	99.59
9	10	95.24	93.50	91.61	80.55	1.09	93.25
9	11	92.01	87.48	84.64	65.75	1.28	87.10

[a] The values of the *A*, *B*, and *C* tranches and the *Z*-bond are expressed as a percentage of their initial face values. The value of the Residual is expressed as a percentage of the initial face value of the collateral.

been outstanding for two years; Panel B gives the values after four years; and Panel C gives the values after six years.[11] To conduct the analysis, it is necessary to make an assumption regarding the fraction of the collateral that has been prepaid. These fractions are assumed to be 0.10, 0.20, and 0.30 in Panels A, B, and C, respectively. The values are expressed as a percentage of the remaining principal balance of the tranche. In Panel C, the *A* tranche has been fully retired, whereas in none of the panels have any of the other tranches begun to be retired. In general, for the standard sequential pay CMO, the sensitivity of the tranches to changes in interest rates declines as the security ages. Of course, the effect of aging will be different for CMOs with different structures. For example, for the CMO with the PAC, the *A* tranche will not be fully retired when 30% of the collateral has been retired. And the effect of aging will be different for a CMO with a sequential pay floater versus a simultaneous pay floater. Further analysis could demonstrate the effects of aging for various CMO structures.

6. Conclusion

This paper presents a model for the valuation of Collateralized Mortgage Obligations (CMOs). A contingent claims valuation approach is used which relies on a two-factor model of interest rates and an empirically estimated prepayment function. The securities being valued are sequential pay fixed-rate tranches, Planned Amortization Class (PAC) tranches, Targeted Amortization Class (TAC) tranches, floating-rate tranches, Interest Only (IO) and Principal Only (PO) tranches, Z-bonds and Residuals. Monte Carlo simulation is used to solve the differential equations governing the values of the derivative MBSs, subject to stochastic terminal conditions. The sensitivities of the value of the different securities to changes in interest rates and to the contractual features of the CMO are studied. In general, the greater the call, or prepayment, risk borne by a tranche, the more sensitive is the value of the tranche to changes in interest rates.

[11] Because the first CMO was issued in 1984, the oldest CMOs have been outstanding for only six years.

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